

REPRESENTATION THEORY
 An Electronic Journal of the American Mathematical Society
 Volume 21, Pages 322–353 (September 15, 2017)
<http://dx.doi.org/10.1090/ert/501>

\mathbf{Z}/m -GRADED LIE ALGEBRAS AND PERVERSE SHEAVES, II

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ABSTRACT. We consider a fixed block for the equivariant perverse sheaves with nilpotent support on the 1-graded component of a semisimple cyclically graded Lie algebra. We give a combinatorial parametrization of the simple objects in that block.

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INTRODUCTION

As in [LY] we fix G , a semisimple simply connected algebraic group over \mathbf{k} (an algebraically closed field of characteristic $p \geq 0$) and a \mathbf{Z}/m -grading $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{g}_i$ for the Lie algebra \mathfrak{g} of G ; here m is an integer > 0 and p is 0 or a large prime number. We fix $\eta \in \mathbf{Z} - \{0\}$ and let δ be the image of η in \mathbf{Z}/m . Let $G_{\underline{0}}$ be the closed connected subgroup of G with Lie algebra $\mathfrak{g}_{\underline{0}}$; this group acts naturally on \mathfrak{g}_{δ} and on $\mathfrak{g}_{\delta}^{nil} = \mathfrak{g}_{\delta} \cap \mathfrak{g}^{nil}$ where \mathfrak{g}^{nil} is the variety of nilpotent elements in \mathfrak{g} . Recall that $\mathcal{I}(\mathfrak{g}_{\delta})$ denotes the set of pairs $(\mathcal{O}, \mathcal{L})$ where \mathcal{O} is a $G_{\underline{0}}$ -orbit on $\mathfrak{g}_{\delta}^{nil}$ and \mathcal{L} is an irreducible $G_{\underline{0}}$ -equivariant local system on \mathcal{O} up to isomorphism. Let \mathfrak{B} be the set of isomorphism classes of simple $G_{\underline{0}}$ -equivariant perverse sheaves on $\mathfrak{g}_{\delta}^{nil}$. Then there is a natural bijection $\mathcal{I}(\mathfrak{g}_{\delta}) \rightarrow \mathfrak{B}$ given by $(\mathcal{O}, \mathcal{L}) \mapsto \mathcal{L}^{\sharp}[\dim \mathcal{O}]$ (for the notation \mathcal{L}^{\sharp} see [LY, 0.11]). In [LY, Theorem 0.6] we have shown that \mathfrak{B} can be naturally decomposed as a disjoint union of blocks ${}^{\xi}\mathfrak{B}$ indexed by the $G_{\underline{0}}$ -conjugacy classes of admissible systems $\xi = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ such that if B, B' are in different blocks then the $G_{\underline{0}}$ -equivariant Ext-groups of B with B' are zero. This generalizes a result of [L4] in the \mathbf{Z} -graded case; its analogue in the ungraded case is the known partition of the set G -equivariant simple perverse sheaves on \mathfrak{g}^{nil} into blocks given by the generalized Springer correspondence.

In this paper we give an (essentially) combinatorial way to parametrize the objects in a fixed block ${}^{\xi}\mathfrak{B}$ of \mathfrak{B} . It is known that in the ungraded case, the objects in

Received by the editors October 12, 2016, and, in revised form, June 23, 2017.

2010 *Mathematics Subject Classification*. Primary 20G99.

The first author was supported by NSF grant DMS-1566618.

The second author was supported by NSF grant DMS-1302071 and the Packard Foundation.

a fixed block are indexed by the irreducible representations of a certain (relative) Weyl group attached to the block. In the \mathbf{Z}/m -graded case we will associate to the block ${}^\xi\mathfrak{B}$ a \mathbf{Q} -vector space \mathbf{E} with a certain finite collection of hyperplanes. The complement of the union of these hyperplanes is naturally a union of finitely many (not necessarily conical) chambers which can be taken to form a basis of a $\mathbf{Q}(v)$ -vector space \mathbf{V}' (here v is an indeterminate). The chambers represent various spiral induction associated to the block. The vector space \mathbf{V}' carries a natural, explicit, sesquilinear form $(\cdot) : \mathbf{V}' \times \mathbf{V}' \rightarrow \mathbf{Q}(v)$ defined in terms of dimensions of Ext-groups between the spiral inductions (see [LY, 6.4]) which correspond to the chambers. We show that the left radical of this form is the same as its right radical. Taking the quotient of \mathbf{V}' by the left or right radical we obtain a $\mathbf{Q}(v)$ -vector space \mathbf{V} with an induced sesquilinear form (\cdot) . It turns out that \mathbf{V} is naturally isomorphic to the Grothendieck group based on ${}^\xi\mathfrak{B}$ (tensored with $\mathbf{Q}(v)$) and, in particular, the number of elements in ${}^\xi\mathfrak{B}$ is equal to $\dim_{\mathbf{Q}(v)} \mathbf{V}$. The vector space \mathbf{V} has a natural bar involution $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$ and a natural \mathcal{A} -lattice $\mathbf{V}_{\mathcal{A}}$ in \mathbf{V} , both defined combinatorially. (Here $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$.) We then define a subset \mathbf{B}' of \mathbf{V} as the set of all $b \in \mathbf{V}_{\mathcal{A}}$ such that $\bar{b} = b$ and $(b : b) \in 1 + v\mathbf{Z}[[v]]$. We show that \mathbf{B}' is a signed basis of \mathbf{V} . (Although the definition of \mathbf{B}' is combinatorial, the proof of the fact that it is a well-defined signed basis is based on geometry; it is not combinatorial. It would be desirable to find a proof without using geometry.) Let \mathbf{B}'/\pm be the set of orbits of the $\mathbf{Z}/2$ -action $b \mapsto -b$ on \mathbf{B}' . We show that \mathbf{B}'/\pm is in natural bijection with the given block ${}^\xi\mathfrak{B}$. Thus \mathbf{B}'/\pm could be regarded as a combinatorial index set for ${}^\xi\mathfrak{B}$. (A similar result in the \mathbf{Z} -graded case appears in [L4].)

We now discuss the contents of the various sections. In Section 10 we define the \mathbf{Q} -vector space \mathbf{E} with its hyperplane arrangement associated to a block. We also define the $\mathbf{Q}(v)$ -vector space \mathbf{V}' with its sesquilinear form (\cdot) , its quotient space \mathbf{V} and the bar operator. In Section 11 we define the \mathcal{A} -lattice $\mathbf{V}_{\mathcal{A}}$ in \mathbf{V} and the signed basis \mathbf{B}' . In Section 12 we prove some purity properties of the cohomology sheaves of the simple G_0 -equivariant perverse sheaves on $\mathfrak{g}_\delta^{nil}$, which generalize those in the \mathbf{Z} -graded case given in [L4]. In Section 13 we generalize an argument in [L4] to express the matrix whose entries are the values of the (\cdot) -pairing at two elements of \mathfrak{B} as a product of three matrices. This is used in Section 14 to prove the vanishing of the odd cohomology sheaves of the intersection cohomology of the closure of any G_0 -orbit in $\mathfrak{g}_\delta^{nil}$ with coefficients in an irreducible G_0 -equivariant local system on that orbit. This generalizes a result in [L4] in the \mathbf{Z} -graded case whose proof was quite different (it was based on geometric arguments which are not available in the present case).

We will adhere to the notation and assumptions of [LY].

10. THE VECTOR SPACE \mathbf{V} AND THE SESQUILINEAR FORM (\cdot)

In this section, we introduce a combinatorial way of calculating the number of irreducible perverse sheaves in a block ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta)$. This is achieved by introducing a finite-dimensional vector space \mathbf{V}' over $\mathbf{Q}(v)$ together with a sesquilinear form on it coming from the pairing between spiral inductions in the fixed block ξ . Then the number of irreducible perverse sheaves in ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta)$ turns out to be the rank of this sesquilinear form on \mathbf{V}' (Proposition 10.19).

10.1. We fix $\xi \in \mathfrak{T}_\eta$ and a representative $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$ for ξ . We also fix $\phi = (e, h, f) \in J^M$ such that $e \in \mathring{\mathfrak{m}}_\eta$, $h \in \mathfrak{m}_0$, $f \in \mathfrak{m}_{-\eta}$. We set $\iota = \iota_\phi \in Y_M$. Let

$$Z = Z_M^0.$$

Since Z is a torus, Y_Z is naturally a free abelian group (with operation written as addition) and $\mathbf{E} := Y_{Z, \mathbf{Q}}$ may be naturally identified with the \mathbf{Q} -vector space $\mathbf{Q} \otimes Y_Z$. Let $X_Z = \text{Hom}(Y_Z, \mathbf{Z})$ and let $\langle \cdot \rangle$ be the obvious perfect bilinear pairing $Y_Z \times X_Z \rightarrow \mathbf{Z}$. This extends to a bilinear pairing $\mathbf{E} \times (\mathbf{Q} \otimes X_Z) \rightarrow \mathbf{Q}$ denoted again by $\langle \cdot \rangle$. For any $\alpha \in X_Z$ let

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g}; \text{Ad}(z)x = \alpha(z)x \quad \forall z \in Z\}.$$

For any $(\alpha, i) \in X_Z \times \mathbf{Z}/m$ let

$$\mathfrak{g}_i^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_i.$$

For any $(\alpha, n, i) \in X_Z \times \mathbf{Z} \times \mathbf{Z}/m$ let

$$\mathfrak{g}_i^{\alpha, n} = \mathfrak{g}_i^\alpha \cap (\iota_n \mathfrak{g}_i).$$

For any $i \in \mathbf{Z}/m$, let $\mathcal{R}_i = \{(\alpha, n) \in X_Z \times \mathbf{Z}; \mathfrak{g}_i^{\alpha, n} \neq 0\}$, $\mathcal{R}_i^* = \{(\alpha, n) \in \mathcal{R}_i; \alpha \neq 0\}$. Note that

(a) $\dim \mathfrak{g}_i^{\alpha, n} = \dim \mathfrak{g}_{-i}^{-\alpha, -n}$ for any α, n, i ; hence $(\alpha, n) \mapsto (-\alpha, -n)$ is a bijection $\mathcal{R}_i \xrightarrow{\sim} \mathcal{R}_{-i}$ and a bijection $\mathcal{R}_i^* \xrightarrow{\sim} \mathcal{R}_{-i}^*$.

We have

$$\mathfrak{g} = \oplus_{(\alpha, n, i) \in X_Z \times \mathbf{Z} \times \mathbf{Z}/m} (\mathfrak{g}_i^{\alpha, n}) = \oplus_{i \in \mathbf{Z}/m, (\alpha, n) \in \mathcal{R}_i} (\mathfrak{g}_i^{\alpha, n}).$$

For any $N \in \mathbf{Z}$ and any $(\alpha, n) \in \mathcal{R}_N^*$ we set

$$\mathfrak{H}_{\alpha, n, N} = \{\varpi \in \mathbf{E}; \langle \varpi : \alpha \rangle = 2N/\eta - n\}.$$

This is an affine hyperplane in \mathbf{E} . We set

$$\mathbf{E}' = \mathbf{E} - \cup_{N \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_N^*} \mathfrak{H}_{\alpha, n, N}.$$

10.2. Recall the notation $Y_{H, \mathbf{Q}}$ for a connected algebraic group H with Lie algebra \mathfrak{h} from [LY, 0.11]. For $\mu, \mu' \in Y_{H, \mathbf{Q}}$, we say μ commutes with μ' if for some r, r' in $\mathbf{Z}_{>0}$ such that $r\mu, r'\mu' \in Y_H$, the images of the homomorphisms $r\mu, r'\mu' : \mathbf{k}^* \rightarrow H$ commute with each other. This property is independent of the choice of r, r' . If μ commutes with μ' , and $r, r' \in \mathbf{Z}_{>0}$ are as above, then letting $\lambda = r\mu, \lambda' = r'\mu'$, we have the homomorphism $\nu : \mathbf{k}^* \rightarrow H$ given by $\nu(t) = \lambda(t^{r'})\lambda'(t^r)$. We then define $\mu + \mu' := \nu/(rr') \in Y_{H, \mathbf{Q}}$. Then $\mu + \mu'$ is independent of the choice of r, r' . Moreover, for any $\kappa \in \mathbf{Q}$, we have

$$(a) \quad \mu + \mu' \big|_{\kappa} \mathfrak{h} = \oplus_{s, t \in \mathbf{Q}; s+t=\kappa} (\mu_s \mathfrak{h} \cap \mu'_t \mathfrak{h}).$$

Let $\varpi \in \mathbf{E}$. Since $\varpi \in \mathbf{E} = Y_{Z, \mathbf{Q}}$ commutes with $\iota \in Y_{M_0}$, by the above discussion, we may define

$$\underline{\varpi} = \frac{|\eta|}{2}(\varpi + \iota) \in Y_{M_0, \mathbf{Q}} \subset Y_{G_0, \mathbf{Q}}.$$

From the definitions, $\underline{\varpi}$ may be computed as follows. We choose $f \in \mathbf{Z}_{>0}$ such that $\lambda' := f\varpi \in Y_Z$ and $f/|\eta| \in \mathbf{Z}$; we define $\lambda \in Y_{G_0}$ by $\lambda(t) = \iota(t^f)\lambda'(t) = \lambda'(t)\iota(t^f)$ for all $t \in \mathbf{k}^*$; we then have $\underline{\varpi} = \frac{|\eta|}{2f}\lambda \in Y_{G_0, \mathbf{Q}}$.

We shall set $\epsilon = \dot{\eta} \in \{1, -1\}$ (the sign of η , see 0.12).

Now $\epsilon \mathfrak{p}_*^{\underline{\varpi}}, \epsilon \tilde{\mathfrak{f}}_*^{\underline{\varpi}}$ are well-defined; see [LY, 2.5, 2.6].

We set

$$\epsilon \tilde{\mathfrak{l}}^{\underline{\varpi}} = \oplus_{N \in \mathbf{Z}} \epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}}.$$

We show:

(b) We have $\epsilon \tilde{\mathfrak{l}}_*^{\underline{\varpi}} = \mathfrak{m}_*$ if and only if $\varpi \in \mathbf{E}'$.

If $x \in \mathfrak{g}_i^{\alpha, n}$ and $t \in \mathbf{k}^*$, then

$$\mathrm{Ad}(\lambda(t))x = \mathrm{Ad}(\iota(t^f)) \mathrm{Ad}(\lambda'(t))x = t^{nf + \langle \lambda' : \alpha \rangle} x$$

and $\mathrm{Ad}(\iota(t))x = t^n x$. Thus, $\mathfrak{g}_i^{\alpha, n} \subset \lambda_{nf + \langle \lambda' : \alpha \rangle} \mathfrak{g}_i$ and $\mathfrak{g}_i^{\alpha, n} \subset \iota_n \mathfrak{g}_i$. It follows that for any $k \in \mathbf{Z}$ we have

$$(c) \quad \lambda_k \mathfrak{g}_i = \oplus_{(\alpha, n) \in \mathcal{R}_i; nf + \langle \lambda' : \alpha \rangle = k} (\mathfrak{g}_i^{\alpha, n}),$$

$$(d) \quad \iota_k \mathfrak{g}_i = \oplus_{(\alpha, n) \in \mathcal{R}_i; n = k} (\mathfrak{g}_i^{\alpha, n}).$$

For any $N \in \mathbf{Z}$ we have $\epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}} = \lambda_{2fN/\eta} \mathfrak{g}_N$. We see that

$$(e) \quad \epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}} = \oplus_{(\alpha, n) \in \mathcal{R}_N; nf + \langle \lambda' : \alpha \rangle = 2fN/\eta} (\mathfrak{g}_N^{\alpha, n}).$$

Recall from 1.2(e) that if $N \in \mathbf{Z}$, $\mathfrak{m}_N \neq 0$, then $N \in \eta\mathbf{Z}$. Let $N \in \eta\mathbf{Z}$. From the arguments in 3.3 (or by [LY, 10.3(a)]) we see that, for some $\varpi' \in \mathbf{E}$, we have $\epsilon \tilde{\mathfrak{l}}_*^{\underline{\varpi}'} = \mathfrak{m}_*$. (Here $\underline{\varpi}'$ is attached to ϖ' in the same way that $\underline{\varpi}$ was attached to ϖ .) Hence, using (e) with ϖ replaced by ϖ' , we see that there exists a subset R_N of \mathcal{R}_N such that $\mathfrak{m}_N = \oplus_{(\alpha, n) \in R_N} (\mathfrak{g}_N^{\alpha, n})$. If $(\alpha, n) \in R_N$, then $0 \neq \mathfrak{g}_N^{\alpha, n} \subset \mathfrak{m}_N$. Hence for $x \in \mathfrak{g}_N^{\alpha, n} - \{0\}$ and $z \in Z$, $t \in \mathbf{k}^*$, we have $\mathrm{Ad}(z)x = \alpha(z)x$, $\mathrm{Ad}(\iota(t))x = t^n x$; but $\mathrm{Ad}(z)x = x$ since Z is in the center of M and $\mathrm{Ad}(\iota(t))x = t^{2N/\eta} x$ since $x \in \mathfrak{m}_N$. Thus $\alpha(z) = 1$ (so that $\alpha = 0$) and $n = 2N/\eta$. We see that $R_N \subset \{(0, 2N/\eta)\}$. Conversely, we show that $\mathfrak{g}_N^{0, 2N/\eta} \subset \mathfrak{m}_N$. If $x \in \mathfrak{g}_N^{0, 2N/\eta}$, then $\mathrm{Ad}(z)x = x$ for all $z \in Z$ and $\mathrm{Ad}(\iota(t))x = t^{2N/\eta} x$ for all $t \in \mathbf{k}^*$. Since ξ in 9.1 is admissible, there exists $t_0 \in \mathbf{k}^*$, $z \in Z$, such that \mathfrak{m} is the fixed point set of $\mathrm{Ad}(\iota(t_0)z)\theta : \mathfrak{g} \rightarrow \mathfrak{g}$. Since \mathfrak{m}_η is contained in this fixed point set and $\mathrm{Ad}(z)$ acts trivially on it, we see that $t_0^2 \zeta^\eta y = y$ for all $y \in \mathfrak{m}_\eta$.

If $\mathfrak{m}_\eta \neq 0$, it follows that $t_0^2 \zeta^\eta = 1$. Thus we have $\mathrm{Ad}(\iota(t)z)\theta(x) = t_0^{2N/\eta} \zeta^N x = (t_0^2 \zeta^\eta)^{N/\eta} x = x$. (Here we use that $N \in \eta\mathbf{Z}$.) Thus x is contained in the fixed point set of $\mathrm{Ad}(\iota(t_0)z)\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, so that $x \in \mathfrak{m}_N$. We see that for any $N \in \eta\mathbf{Z}$ we have

$$(f) \quad \mathfrak{m}_N = \mathfrak{g}_N^{0, 2N/\eta}.$$

Now (f) also holds when $\mathfrak{m}_\eta = 0$. (In that case we must have $\iota = 1$ hence $\mathfrak{m} = \mathfrak{m}_0$, so that $\mathfrak{m}_N = 0$ for $N \neq 0$. Moreover, by [LY, 3.6(d)], \mathfrak{m} is a Cartan subalgebra of \mathfrak{g}_0 and $Z = M$. We also have $\mathfrak{g}_N^{0, 2N/\eta} = 0$ for $N \neq 0$. Thus, for $N \neq 0$, (f) states that $0 = 0$. If $N = 0$, (f) states that \mathfrak{m} is its own centralizer in \mathfrak{g}_0 , which is clear.)

From (e), (f) we see that $\mathfrak{m}_N \subset \epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}}$ for all $N \in \eta\mathbf{Z}$ and that

(g) for $N \in \eta\mathbf{Z}$ we have $\mathfrak{m}_N = \epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}}$ if and only if the following holds: $\{(\alpha, n) \in \mathcal{R}_N; n + \langle \varpi : \alpha \rangle = 2N/\eta\}$ is equal to $\{(0, 2N/\eta)\}$ if $(0, 2N/\eta) \in \mathcal{R}_N$ and is empty if $(0, 2N/\eta) \notin \mathcal{R}_N$;

(g') for $N \in \mathbf{Z} - \eta\mathbf{Z}$ we have $\mathfrak{m}_N = \epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}}$ (or equivalently $\epsilon \tilde{\mathfrak{l}}_N^{\underline{\varpi}} = 0$) if and only if $\{(\alpha, n) \in \mathcal{R}_N; n + \langle \varpi : \alpha \rangle = 2N/\eta\} = \emptyset$. The condition in (g) can be also expressed

as follows:

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}^*; n + \langle \varpi : \alpha \rangle = 2N/\eta\} = \emptyset \text{ for any } N \in \eta\mathbf{Z}.$$

The condition in (\mathfrak{g}') can be also expressed as follows:

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n + \langle \varpi : \alpha \rangle = 2N/\eta\} = \emptyset \text{ for any } N \in \mathbf{Z} - \eta\mathbf{Z}.$$

Indeed, it is enough to show that if $N \in \mathbf{Z} - \eta\mathbf{Z}$ and $\{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n + \langle \varpi : \alpha \rangle = 2N/\eta\}$, then we have automatically $\alpha \neq 0$. (Assume that $\alpha = 0$. Then $n = 2N/\eta$ so that n is an odd integer. Since $\mathfrak{g}^0 = \mathfrak{m}$ and $\mathfrak{g}_{\underline{N}}^{0,n} \neq 0$ we see that ${}^{\iota}_n \mathfrak{m} \neq 0$. Using 1.2(d) we deduce that n is even, a contradiction.)

We see that (b) holds.

From (c) we deduce that

$${}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi} = \oplus_{k \in \mathbf{Z}; k \geq 2fN/\eta} ({}^{\lambda}_k \mathfrak{g}_{\underline{N}}) = \oplus_{k \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_{\underline{N}}; k \geq 2fN/\eta, nf + \langle \lambda' : \alpha \rangle = k} (\mathfrak{g}_{\underline{N}}^{\alpha, n}),$$

hence

$$(h) \quad {}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi} = \oplus_{(\alpha, n) \in \mathcal{R}_{\underline{N}}; \langle \varpi : \alpha \rangle \geq 2N/\eta - n} (\mathfrak{g}_{\underline{N}}^{\alpha, n}).$$

The nilradical ${}^{\epsilon} \mathfrak{u}_{*}^{\varpi}$ of ${}^{\epsilon} \mathfrak{p}_{*}^{\varpi}$ is given by

$$(i) \quad {}^{\epsilon} \mathfrak{u}_{\underline{N}}^{\varpi} = \oplus_{(\alpha, n) \in \mathcal{R}_{\underline{N}}^*; \langle \varpi : \alpha \rangle > 2N/\eta - n} (\mathfrak{g}_{\underline{N}}^{\alpha, n}).$$

We see that for ϖ, ϖ' in \mathbf{E} and $N \in \mathbf{Z}$, the following two conditions are equivalent:

$$(I) \quad {}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi} = {}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi'}.$$

$$(II) \quad \text{For any } (\alpha, n) \in \mathcal{R}_{\underline{N}}^* \text{ we have } \langle \varpi : \alpha \rangle + n \geq 2N/\eta \Leftrightarrow \langle \varpi' : \alpha \rangle + n \geq 2N/\eta.$$

For ϖ, ϖ' in \mathbf{E}' we say that $\varpi \equiv \varpi'$ if for any $N \in \mathbf{Z}$ and any $(\alpha, n) \in \mathcal{R}_{\underline{N}}^*$ we have

$$(\langle \varpi : \alpha \rangle + n - 2N/\eta)(\langle \varpi' : \alpha \rangle + n - 2N/\eta) > 0.$$

This is clearly an equivalence relation on \mathbf{E}' . From the equivalence of (I), (II) above, we see that

$$(j) \quad \varpi \equiv \varpi' \Leftrightarrow {}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi} = {}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi'} \quad \forall N \in \mathbf{Z}.$$

For any $\varpi \in \mathbf{E}'$ we set

$$(k) \quad \begin{aligned} I_{\varpi} &= {}^{\epsilon} \text{Ind}_{\epsilon \mathfrak{p}_{\underline{\eta}}^{\varpi}}^{\mathfrak{g}_{\delta}} (\tilde{C}[-\dim \mathfrak{m}_{\eta}]) \in \mathcal{Q}(\mathfrak{g}_{\delta}), \\ \tilde{I}_{\varpi} &= {}^{\epsilon} \widetilde{\text{Ind}}_{\epsilon \mathfrak{p}_{\underline{\eta}}^{\varpi}}^{\mathfrak{g}_{\delta}} (\tilde{C}) \in \mathcal{Q}(\mathfrak{g}_{\delta}). \end{aligned}$$

Here we regard \tilde{C} as an object of $\mathcal{Q}(\tilde{\mathfrak{l}}_{\underline{\eta}}^{\varpi}) = \mathcal{Q}(\mathfrak{m}_{\eta})$, see (b). Note that in $\mathcal{K}(\mathfrak{g}_{\delta})$ we have

$$\tilde{I}_{\varpi} = v^{h(\varpi)} I_{\varpi},$$

where

$$h(\varpi) = \dim {}^{\epsilon} \mathfrak{u}_0^{\varpi} + \dim {}^{\epsilon} \mathfrak{u}_{\underline{\eta}}^{\varpi} + \dim \mathfrak{m}_{\eta} = \dim {}^{\epsilon} \mathfrak{u}_0^{\varpi} + \dim {}^{\epsilon} \mathfrak{p}_{\underline{\eta}}^{\varpi}.$$

We show:

$$(l) \quad \text{If } \varpi, \varpi' \in \mathbf{E}', \varpi \equiv \varpi', \text{ then } I_{\varpi} = I_{\varpi'}, h(\varpi) = h(\varpi').$$

Indeed, in this case we have ${}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi} = {}^{\epsilon} \mathfrak{p}_{\underline{N}}^{\varpi'}$ for all $N \in \mathbf{Z}$ (see (j)) and the result follows from the definitions.

10.3. We keep the setup of 10.1, 10.2. As in [LY, 2.9], for any $N \in \mathbf{Z}$ we set

$$\tilde{\mathfrak{l}}_N^\phi = {}^{\iota}_{2N/\eta} \mathfrak{g}_N \text{ if } 2N/\eta \in \mathbf{Z}, \quad \tilde{\mathfrak{l}}_N^\phi = 0 \text{ if } 2N/\eta \notin \mathbf{Z}.$$

Hence

$$(a) \quad \tilde{\mathfrak{l}}_N^\phi = \bigoplus_{(\alpha, n) \in \mathcal{R}_N; n=2N/\eta} \mathfrak{g}_N^{\alpha, n}.$$

We set $\tilde{\mathfrak{l}}^\phi = \bigoplus_{N \in \mathbf{Z}} \tilde{\mathfrak{l}}_N^\phi$.

Let

$$\mathbf{E}'' = \mathbf{E} - \bigcup_{N \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_N^*; n \neq 2N/\eta} \mathfrak{H}_{\alpha, n, N}.$$

For $\varpi \in \mathbf{E}$ we show:

(b) *We have ${}^\epsilon \tilde{\mathfrak{l}}^\varpi \subset \tilde{\mathfrak{l}}^\phi$ if and only if $\varpi \in \mathbf{E}''$.*

Using (a) and 10.2(e) we see that we have $\bigoplus_N {}^\epsilon \tilde{\mathfrak{l}}_N^\varpi \subset \tilde{\mathfrak{l}}^\phi$ if and only if for any $N \in \mathbf{Z}$ we have

$$\{(\alpha, n) \in \mathcal{R}_N; n + \langle \varpi : \alpha \rangle = 2N/\eta\} \subset \{(\alpha, n) \in \mathcal{R}_N; n = 2N/\eta\},$$

or equivalently,

$$\{(\alpha, n) \in \mathcal{R}_N; n + \langle \varpi : \alpha \rangle = 2N/\eta; n \neq 2N/\eta\} = \emptyset,$$

or equivalently,

$$\{(\alpha, n) \in \mathcal{R}_N^*; \langle \varpi : \alpha \rangle = 2N/\eta - n \neq 0\} = \emptyset.$$

This is the same as the condition that $\varpi \in \mathbf{E}''$. This proves (b).

10.4. We show:

(a) *Let \mathfrak{p}_* be an ϵ -spiral with a splitting \mathfrak{m}'_* such that \mathfrak{m} is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{m}' = \bigoplus_N \mathfrak{m}'_N$ compatible with the \mathbf{Z} -gradings. Then for some $\varpi \in \mathbf{E}$ we have $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\varpi$, $\mathfrak{m}'_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\varpi$.*

We can find $\mu \in Y_{G_{\underline{0}}, \mathbf{Q}}$ such that $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\mu$ and $\mathfrak{m}'_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\mu$. Let $M'_0 = e^{\mathfrak{m}'_0}$. We choose $f \in \mathbf{Z}_{>0}$ such that $\lambda_1 := f\mu \in Y_{G_{\underline{0}}}$. We have $\mathfrak{m}'_N = {}^{\lambda_1}_{fN\epsilon} \mathfrak{g}_N$ for all $N \in \mathbf{Z}$. Let $N \in \eta\mathbf{Z}$. Since $\mathfrak{m}_N \subset \mathfrak{m}'_N$, for any $x \in \mathfrak{m}_N$ and any $t \in \mathbf{k}^*$ we have $\text{Ad}(\lambda_1(t))x = t^{fN\epsilon}x$; we have also $\text{Ad}(\iota(t))x = t^{2N/\eta}x$.

Hence $\text{Ad}(\lambda_1(t^2)\iota(t^{-f|\eta|}))x = x$ for any $x \in \mathfrak{m}_N$. Since this holds for any $N \in \eta\mathbf{Z}$, it follows that the image of $\lambda_1^2 \iota^{-f|\eta|} : \mathbf{k}^* \rightarrow M'_0$ commutes with M . Since M is a Levi subgroup of a parabolic subgroup of M' , the image of $\lambda_1^2 \iota^{-f|\eta|}$ is contained in Z_M , hence in $Z_M^0 = Z$. In particular, the images of ι and $\lambda_1^2 \iota^{-f|\eta|}$ commute with each other, hence the images of λ_1 and ι commute with each other. It therefore makes sense to write $\lambda' := 2\lambda_1 - f|\eta|\iota \in Y_M$ and we actually have $\lambda' \in Y_Z$. Let $\varpi = |\eta|^{-1}f^{-1}\lambda' \in Y_{Z, \mathbf{Q}} = \mathbf{E}$. We have $\varpi = |\eta|^{-1}(2\mu - |\eta|\iota)$ hence $\varpi + \iota = 2|\eta|^{-1}\mu$, that is, $\mu = \underline{\varpi}$ and (a) is proved.

Let \mathcal{C}' be the collection of ϵ -spirals \mathfrak{p}_* such that \mathfrak{m}_* is a splitting of \mathfrak{p}_* . We show:

(b) *\mathcal{C}' coincides with the collection of ϵ -spirals of the form ${}^\epsilon \mathfrak{p}_*^\varpi$ with $\varpi \in \mathbf{E}'$.*

Assume first that $\mathfrak{p}_* \in \mathcal{C}'$. Using (a) with $\mathfrak{m}'_* = \mathfrak{m}_*$ we see that for some $\varpi \in \mathbf{E}$ we have $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\varpi$, $\mathfrak{m}_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\varpi$. Using 10.2(b), we see that $\varpi \in \mathbf{E}'$.

Conversely, assume that $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\varpi$ for some $\varpi \in \mathbf{E}'$. From 10.2(b) we have $\mathfrak{m}_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\varpi$. Thus $\mathfrak{p}_* \in \mathcal{C}'$. This proves (b).

Let \mathcal{C}'' be the collection of ϵ -spirals \mathfrak{p}_* with the following property: there exists a splitting \mathfrak{m}'_* of \mathfrak{p}_* such that $\mathfrak{m}_N \subset \mathfrak{m}'_N \subset \tilde{\mathfrak{l}}_N^\phi$ for all N . We show:

(c) \mathcal{C}'' coincides with the collection of ϵ -spirals of the form ${}^\epsilon\mathfrak{p}_*^{\varpi}$ with $\varpi \in \mathbf{E}''$.

Assume first that $\mathfrak{p}_* \in \mathcal{C}''$ and let \mathfrak{m}'_* be a splitting of \mathfrak{p}_* as in the definition of \mathcal{C}'' . Now \mathfrak{m} is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{m}' = \bigoplus_N \mathfrak{m}'_N$ compatible with the \mathbf{Z} -gradings of \mathfrak{m} and \mathfrak{m}' . (Indeed, from the proof of 3.7(c) we see that there exists $\lambda \in Y_Z$ such that $\mathfrak{m} = \{y \in \tilde{\mathfrak{l}}^\phi; \text{Ad}(\lambda(t))y = y \quad \forall t \in \mathbf{k}^*\}$. Since $\mathfrak{m}' \subset \tilde{\mathfrak{l}}^\phi$ we see that $\mathfrak{m} = \{y \in \mathfrak{m}'; \text{Ad}(\lambda(t))y = y \quad \forall t \in \mathbf{k}^*\}$, as required.) Using (a), we see that for some $\varpi \in \mathbf{E}$ we have $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^{\varpi}$, $\mathfrak{m}'_* = {}^\epsilon\tilde{\mathfrak{l}}_*^{\varpi}$. Since ${}^\epsilon\tilde{\mathfrak{l}}_N^{\varpi} \subset \tilde{\mathfrak{l}}_N^\phi$ for all N , we see from 10.3(b) that $\varpi \in \mathbf{E}''$.

Conversely, assume that $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^{\varpi}$ for some $\varpi \in \mathbf{E}''$. Let $\mathfrak{m}'_* = {}^\epsilon\tilde{\mathfrak{l}}_*^{\varpi}$. From 10.3(b) we see that ${}^\epsilon\tilde{\mathfrak{l}}_N^{\varpi} \subset \tilde{\mathfrak{l}}_N^\phi$ for all N . We also have $\mathfrak{m}_N \subset {}^\epsilon\tilde{\mathfrak{l}}_N^{\varpi}$ for all N . Thus $\mathfrak{p}_* \in \mathcal{C}''$. This proves (c).

Note that:

(d) If $\mathfrak{p}_* \in \mathcal{C}''$, then the splitting \mathfrak{m}'_* in the definition of \mathcal{C}'' is in fact unique.

Indeed, assume that $\mathfrak{m}'_*, \tilde{\mathfrak{m}}'_*$ are splittings of \mathfrak{p}_* such that $\mathfrak{m}_N \subset \mathfrak{m}'_N$, $\mathfrak{m}_N \subset \tilde{\mathfrak{m}}'_N$ for all N . By [LY, 2.7] we can find $u \in U_0$ (U_0 as in [LY, 2.5]) such that $\text{Ad}(u)\mathfrak{m}'_* = \tilde{\mathfrak{m}}'_*$. In particular, we have $\text{Ad}(u)\mathfrak{m}'_0 = \tilde{\mathfrak{m}}'_0$. This implies $u = 1$ since $\mathfrak{m}'_0, \tilde{\mathfrak{m}}'_0$ are both Levi subalgebras of \mathfrak{p}_0 containing \mathfrak{m}_0 . Hence $\mathfrak{m}'_* = \tilde{\mathfrak{m}}'_*$.

10.5. Let $\mathfrak{p}_*, \mathfrak{p}'_*$ be two ϵ -spirals such that $\mathfrak{p}_N \subset \mathfrak{p}'_N$ for all N . Let $\mathfrak{u}_*, \mathfrak{u}'_*$ be their nilradicals. Then we have $\mathfrak{u}'_N \subset \mathfrak{u}_N$ for all N . In particular, $\mathfrak{u}'_N \subset \mathfrak{p}_N$ for all N . We show:

(a) $\bigoplus_N \mathfrak{p}_N / \mathfrak{u}'_N$ is a parabolic subalgebra of $\mathfrak{l}' = \bigoplus_N \mathfrak{p}'_N / \mathfrak{u}'_N$.

Let $P_0 = e^{\mathfrak{p}_0}$, $P'_0 = e^{\mathfrak{p}'_0}$ and let $U_0 = U_{P_0}$, $U'_0 = U_{P'_0}$. We have $P_0 \subset P'_0$, $U'_0 \subset U_0$. Now $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^\mu$ and $\mathfrak{p}'_* = {}^\epsilon\mathfrak{p}_*^{\mu'}$ for some $\mu = \lambda/r$, $\mu' = \lambda'/r'$, with $\lambda, \lambda' \in Y_{G_0}$ and $r, r' \in \mathbf{Z}_{>0}$. We have $\lambda(\mathbf{k}^*) \subset P_0$, $\lambda'(\mathbf{k}^*) \subset P'_0$. We can find Levi subgroups $\tilde{L}_0, \tilde{L}'_0$ of P_0, P'_0 such that $\tilde{L}_0 \subset \tilde{L}'_0$. By conjugating λ (resp. λ') by an element of U_0 (resp. U'_0) we can assume that $\lambda \in \mathcal{Z}_{\tilde{L}_0}$, $\lambda' \in \mathcal{Z}_{\tilde{L}'_0}$. Since $\mathcal{Z}_{\tilde{L}'_0} \subset \mathcal{Z}_{\tilde{L}_0}$, we have $\lambda(t)\lambda'(t') = \lambda'(t')\lambda(t)$ for any t, t' in \mathbf{k}^* . Hence we have $\mathfrak{g} = \bigoplus_{k, k' \in \mathbf{Q}, i \in \mathbf{Z}/m} (\frac{\mu, \mu'}{k, k'} \mathfrak{g}_i)$, where $\frac{\mu, \mu'}{k, k'} \mathfrak{g}_i = \frac{\mu}{k} \mathfrak{g}_i \cap \frac{\mu'}{k'} \mathfrak{g}_i$. We have

$$\mathfrak{p}_N = \bigoplus_{k, k' \in \mathbf{Q}; k \geq N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right),$$

$$\mathfrak{p}'_N = \bigoplus_{k, k' \in \mathbf{Q}; k' \geq N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right),$$

$$\mathfrak{u}'_N = \bigoplus_{k, k' \in \mathbf{Q}; k' > N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right).$$

Since $\mathfrak{u}'_N \subset \mathfrak{p}_N \subset \mathfrak{p}'_N$ we see that

$$\mathfrak{p}_N = \bigoplus_{k, k' \in \mathbf{Q}; k \geq N, k' \geq N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right),$$

$$\mathfrak{u}'_N = \bigoplus_{k, k' \in \mathbf{Q}; k \geq N, k' > N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right),$$

hence

$$\mathfrak{p}_N / \mathfrak{u}'_N \cong \bigoplus_{k, k' \in \mathbf{Q}; k \geq N, k' = N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right).$$

This is a subspace of

$$\mathfrak{p}'_N / \mathfrak{u}'_N \cong \tilde{\mathfrak{l}}'_N := \bigoplus_{k, k' \in \mathbf{Q}; k' = N} \epsilon \left(\frac{\mu, \mu'}{k, k'} \mathfrak{g}_N \right).$$

Since μ and μ' commute with each other, it makes sense to define $\nu = \mu - \mu' \in Y_{\tilde{L}'_0, \mathbf{Q}}$. Let $\tilde{\nu}' = \oplus_N \tilde{\nu}'_N \subset \mathfrak{g}$. Then \tilde{L}'_0 acts on ν' by the Ad-action and ν induces a \mathbf{Q} -grading $\nu' = \oplus_{k_1 \in \mathbf{Q}} \nu'_{k_1}$. From the definitions we see that $\oplus_N \mathfrak{p}'_N / \mathfrak{u}_N = \oplus_{k_1 \in \mathbf{Q}; k_1 \geq 0} (\nu'_{k_1})$. Thus (a) holds.

A similar argument shows:

(b) If \mathfrak{m}_* is a splitting of \mathfrak{p}_* , the obvious map $\mathfrak{m} = \oplus_N \mathfrak{m}_N \rightarrow \oplus_N \mathfrak{p}_N / \mathfrak{u}'_N$ defines an isomorphism of \mathfrak{m} onto a Levi subalgebra of $\oplus_N \mathfrak{p}_N / \mathfrak{u}'_N$.

10.6. Let ϖ, ϖ' in \mathbf{E}' . For any $t \in \mathbf{Q}$ such that $0 \leq t \leq 1$ we set

$$\varpi_t := t\varpi + (1-t)\varpi'.$$

We assume that there is a unique hyperplane \mathfrak{H} of the form $\mathfrak{H} = \mathfrak{H}_{\alpha_0, n_0, N_0}$ for some (α_0, n_0, N_0) with $N_0 \in \mathbf{Z}$, $(\alpha_0, n_0) \in \mathcal{R}_{N_0}^*$ such that $\varpi_t \in \mathfrak{H}$ for some $t = s$; this s is necessarily unique since $\varpi_0 \notin \mathfrak{H}$. Note, however, that the triple (α_0, n_0, N_0) is not uniquely determined by \mathfrak{H} . We set $\varpi'' = \varpi_s$,

$$\begin{aligned} \mathfrak{p}_* &= {}^\epsilon \mathfrak{p}_*^{\varpi}, \mathfrak{p}'_* = {}^\epsilon \mathfrak{p}_*^{\varpi'}, \mathfrak{u}_* = {}^\epsilon \mathfrak{u}_*^{\varpi}, \mathfrak{u}'_* = {}^\epsilon \mathfrak{u}_*^{\varpi'}, \\ \mathfrak{p}''_* &= {}^\epsilon \mathfrak{p}_*^{\varpi''}, \tilde{\nu}''_* = {}^\epsilon \tilde{\nu}_*^{\varpi''}, \tilde{\nu}'' = \oplus_{N \in \mathbf{Z}} \tilde{\nu}''_N, \\ P_0 &= e^{\mathfrak{p}_0}, P'_0 = e^{\mathfrak{p}'_0}, P''_0 = e^{\mathfrak{p}''_0}. \end{aligned}$$

We show:

(a) For any $N \in \mathbf{Z}$ we have $\mathfrak{p}_N \subset \mathfrak{p}''_N$; hence $P_0 \subset P''_0$. For any $N \in \mathbf{Z}$ we have $\mathfrak{p}'_N \subset \mathfrak{p}''_N$; hence $P'_0 \subset P''_0$.

Using 10.2(h), we see that to prove the first sentence in (a) it is enough to show that for any $(\alpha, n) \in \mathcal{R}_N^*$ such that $\langle \varpi : \alpha \rangle \geq 2N/\eta - n$ we have also $\langle \varpi'' : \alpha \rangle \geq 2N/\eta - n$. (Since $\varpi \in \mathbf{E}'$, we must have $\langle \varpi : \alpha \rangle > 2N/\eta - n$. Since for $t \in \mathbf{Q}$, $s < t \leq 1$ we have $\varpi_t \in \mathbf{E}'$, it follows that for all such t we have $\langle \varpi_t : \alpha \rangle > 2N/\eta - n$. Taking the limit as $t \mapsto s$ we obtain $\langle \varpi'' : \alpha \rangle \geq 2N/\eta - n$, as required). The second sentence in (a) is proved in a similar way.

We show:

(b) Fix $N \in \mathbf{Z}$. If $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, N}$ for any $(\alpha, n) \in \mathcal{R}_N^*$, then we have $\mathfrak{p}_N = \mathfrak{p}''_N = \mathfrak{p}'_N$.

We first show that $\mathfrak{p}_N = \mathfrak{p}''_N$. By the equivalence of (I),(II) in 10.2, it is enough to show that for any $(\alpha, n) \in \mathcal{R}_N^*$ we have

$$\langle \varpi : \alpha \rangle + n - 2N/\eta > 0 \Leftrightarrow \langle \varpi'' : \alpha \rangle + n - 2N/\eta > 0,$$

or equivalently, that $c_1 c_s > 0$, where $c_t = \langle \varpi_t : \alpha \rangle + n - 2N/\eta$ for $t \in \mathbf{Q}$ such that $0 \leq t \leq 1$. From our assumptions we see that $c_t \neq 0$ for all t . It follows that either $c_t > 0$ for all t or $c_t < 0$ for all t . In particular, $c_1 c_s > 0$, as required. The proof of the equality $\mathfrak{p}'_N = \mathfrak{p}''_N$ is entirely similar.

We show:

(c) If $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, 0}$ for any $(\alpha, n) \in \mathcal{R}_0^*$ and $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, \eta}$ for any $(\alpha, n) \in \mathcal{R}_\delta^*$, then $I_\varpi = I_{\varpi'}$, $h(\varpi) = h(\varpi')$.

Indeed, in this case, by (b) we have $\mathfrak{p}_N = \mathfrak{p}'_N$ for $N \in \{0, \eta\}$ and the equality $I_\varpi = I_{\varpi'}$ follows from the definitions. From $\mathfrak{p}_0 = \mathfrak{p}'_0$ we deduce that $\mathfrak{u}_0 = \mathfrak{u}'_0$. Using this and $\mathfrak{p}_\eta = \mathfrak{p}'_\eta$ we deduce that $h(\varpi) = h(\varpi')$. This proves (c).

We show:

(d) If $\mathfrak{H} = \mathfrak{H}_{\alpha_0, n_0, 0}$ for some $(\alpha_0, n_0) \in \mathcal{R}_0^*$ but $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, \eta}$ for any $(\alpha, n) \in \mathcal{R}_\delta^*$, then $I_\varpi \cong I_{\varpi'}$, $h(\varpi) = h(\varpi')$.

By (a) we have $\mathfrak{p}_N \subset \mathfrak{p}_N'', \mathfrak{p}'_N \subset \mathfrak{p}_N''$ for all $N \in \mathbf{Z}$ and by (b) we have $\mathfrak{p}_\eta = \mathfrak{p}_\eta'' = \mathfrak{p}'_\eta$. We can now apply [LY, 4.5(b)] twice to conclude that

$$I_\varpi \cong \bigoplus_{j \in J} I_{\varpi''}[-2a_j], I_{\varpi'} \cong \bigoplus_{j' \in J'} I_{\varpi''}[-2a'_{j'}],$$

where a_j (resp. $a'_{j'}$) are integers such that

$$\rho_{P_0''/P_0!} \bar{\mathbf{Q}}_l = \bigoplus_{j \in J} \bar{\mathbf{Q}}_l[-2a_j], \rho_{P_0''/P_0'!} \bar{\mathbf{Q}}_l = \bigoplus_{j' \in J'} \bar{\mathbf{Q}}_l[-2a'_{j'}].$$

To show that $I_\varpi \cong I_{\varpi'}$ it is then enough to show that

$$\bigoplus_{j \in J} \bar{\mathbf{Q}}_l[-2a_j] \cong \bigoplus_{j' \in J'} \bar{\mathbf{Q}}_l[-2a'_{j'}]$$

or that $\rho_{P_0''/P_0!} \bar{\mathbf{Q}}_l \cong \rho_{P_0''/P_0'!} \bar{\mathbf{Q}}_l$. This is clear since $P_0''/P_0, P_0''/P_0'$ are partial flag manifolds of the reductive quotient L_0'' of P_0'' with respect to two associate parabolic subgroups. The above argument shows also that $\dim U_{P_0} = \dim U_{P_0'}$ hence $\dim \mathfrak{u}_0 = \dim \mathfrak{u}'_0$. Moreover, from (b) we have $\mathfrak{p}_\eta = \mathfrak{p}'_\eta$; we see that $h(\varpi) = h(\varpi')$. This proves (d).

For $N \in \mathbf{Z}$ let \mathfrak{q}_N (resp. \mathfrak{q}'_N) be the image of \mathfrak{p}_N (resp. \mathfrak{p}'_N) under the obvious projection $\mathfrak{p}_N'' \rightarrow \tilde{l}_N''$. From 10.5(a),(b), we see that $\mathfrak{q} = \bigoplus_N \mathfrak{q}_N, \mathfrak{q}' = \bigoplus_N \mathfrak{q}'_N$ are parabolic subalgebras of \tilde{l}'' and \mathfrak{m} is a Levi subalgebra of both \mathfrak{q} and \mathfrak{q}' . We show:

(e) *If $\mathfrak{H} = \mathfrak{H}_{\alpha_0, 2, \eta}$ for some $(\alpha_0, 2) \in \mathcal{R}_\delta^*$, then $I_\varpi \cong I_{\varpi'}$ and $h(\varpi) = h(\varpi')$.*

By (a) we have $\mathfrak{p}_N \subset \mathfrak{p}_N'', \mathfrak{p}'_N \subset \mathfrak{p}_N''$ for all $N \in \mathbf{Z}$. Since $\mathfrak{H}_{\alpha_0, 2, \eta}$ is the unique hyperplane (as in 10.1) on which ϖ'' lies, we have $\varpi'' \in \mathbf{E}''$. Hence by 10.3(b) we have $\tilde{l}_N'' \subset \tilde{l}_N^\phi$ for all $N \in \mathbf{Z}$. In particular, the \mathbf{Z} -grading of \tilde{l}'' is η -rigid and $e \in \tilde{l}_\eta''$, so that $\mathring{\mathfrak{m}}_\eta \subset \mathring{\tilde{l}}_\eta''$. Let $A \in \mathcal{Q}(\tilde{l}_\eta'')$ be the simple perverse sheaf on \tilde{l}_η'' such that the support of A is \tilde{l}_η'' and $A|_{\mathring{\mathfrak{m}}_\eta}$ is equal up to shift to $\tilde{C}|_{\mathring{\mathfrak{m}}_\eta}$. Applying (twice) [LY, 1.8(b)] and the transitivity formula 4.2(a) we deduce

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}) = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta''}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}_\eta}^{\tilde{l}_\eta''}(\tilde{C})) \cong \bigoplus_j {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta''}^{\mathfrak{g}_\delta}(A)[-2s_j][\dim \mathfrak{m}_\eta - \dim \tilde{l}_\eta''],$$

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta'}^{\mathfrak{g}_\delta}(\tilde{C}) = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta''}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}'_\eta}^{\tilde{l}_\eta''}(\tilde{C})) \cong \bigoplus_j {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta''}^{\mathfrak{g}_\delta}(A)[-2s_j][\dim \mathfrak{m}_\eta - \dim \tilde{l}_\eta''],$$

where (s_j) is a certain finite collection of integers. It follows that

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}) \cong {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta'}^{\mathfrak{g}_\delta}(\tilde{C}).$$

This shows that $I_\varpi, I_{\varpi'}$ are isomorphic. It remains to show that $h(\varpi) = h(\varpi')$, for which it suffices to show the following two equalities:

$$(f) \quad \dim \mathfrak{u}_0 = \dim \mathfrak{u}'_0,$$

$$(g) \quad \dim \mathfrak{u}_\eta = \dim \mathfrak{u}'_\eta.$$

Since \mathfrak{p}_0 and \mathfrak{p}'_0 are associate parabolic subalgebras of \mathfrak{p}_0'' sharing the same Levi subalgebra \mathfrak{m}_0 , their unipotent radicals \mathfrak{u}_0 and \mathfrak{u}'_0 have the same dimension. This proves (f).

Now we show (g). By 10.2(i), we have

$$\dim \mathfrak{u}_\eta - \dim \mathfrak{u}'_\eta = \sum_{\alpha \in X_Z; \langle \varpi: \alpha \rangle > 0, \langle \varpi': \alpha \rangle < 0} \dim \mathfrak{g}_\delta^{\alpha, 2} - \sum_{\alpha \in X_Z; \langle \varpi: \alpha \rangle < 0, \langle \varpi': \alpha \rangle > 0} \dim \mathfrak{g}_\delta^{\alpha, 2}.$$

We want to show the above difference is zero. For this, it suffices to show that $\dim \mathfrak{g}_\delta^{\alpha, 2} = \dim \mathfrak{g}_\delta^{-\alpha, 2}$ for any $\alpha \in X_Z$. Note that $\mathfrak{g}_\delta^{\alpha, 2} = \tilde{l}_\eta^\phi \cap \mathfrak{g}^\alpha$, which is the α -weight space of Z on \tilde{l}_η^ϕ . By the property of the \mathfrak{sl}_2 -action on \tilde{l}^ϕ given by the

triple $\phi = (e, h, f)$, we have that $\text{ad}(e)$ induces an isomorphism $\tilde{l}_{-\eta}^{\phi} \cong \tilde{l}_{\eta}^{\phi}$. Since Z commutes with e , this isomorphism restricts to an isomorphism of α -weight spaces $\tilde{l}_{-\eta}^{\phi} \cap \mathfrak{g}^{\alpha} \cong \tilde{l}_{\eta}^{\phi} \cap \mathfrak{g}^{\alpha}$, that is, $\mathfrak{g}_{-\delta}^{\alpha, -2} \cong \mathfrak{g}_{\delta}^{\alpha, 2}$. Therefore $\dim \mathfrak{g}_{\delta}^{\alpha, 2} = \dim \mathfrak{g}_{-\delta}^{\alpha, -2} = \dim \mathfrak{g}_{\delta}^{-\alpha, 2}$, as desired. This proves (g) and finishes the proof of (e).

10.7. Let

$$\begin{aligned} \mathring{\mathbf{E}} &= \mathbf{E} - \cup_{(\alpha, n) \in \mathcal{R}_{\delta}^*; n \neq 2} \mathfrak{H}_{\alpha, n, \eta} \\ &= \{\varpi \in \mathbf{E}; \langle \varpi : \alpha \rangle + n - 2 \neq 0 \quad \forall (\alpha, n) \in \mathcal{R}_{\delta}^* \text{ such that } n \neq 2\}. \end{aligned}$$

Note that $\mathbf{E}' \subset \mathbf{E}'' \subset \mathring{\mathbf{E}}$. For $'\varpi, ''\varpi$ in $\mathring{\mathbf{E}}$ we say that $'\varpi \sim ''\varpi$ if for any $(\alpha, n) \in \mathcal{R}_{\delta}^*$ such that $n \neq 2$ we have

$$(\langle '\varpi : \alpha \rangle + n - 2)(\langle ''\varpi : \alpha \rangle + n - 2) > 0.$$

This is an equivalence relation on $\mathring{\mathbf{E}}$. We show:

(a) Assume that $'\varpi, ''\varpi$ in \mathbf{E}' satisfy $'\varpi \sim ''\varpi$. Then $I_{'\varpi} \cong I_{''\varpi}$ and $h(' \varpi) = h('' \varpi)$; hence $\tilde{I}_{'\varpi} \cong \tilde{I}_{''\varpi}$.

By a known property of hyperplane arrangements, we can find a sequence $\varpi_0, \varpi_1, \dots, \varpi_k$ in \mathbf{E}' such that $\varpi_0 = '\varpi, \varpi_k = ''\varpi$ and such that for any $j \in \{0, 1, \dots, k-1\}$ one of the following holds:

- $\varpi_j \equiv \varpi_{j+1}$;
- $\varpi_j = \varpi, \varpi_{j+1} = \varpi'$ are as in 10.6(c);
- $\varpi_j = \varpi, \varpi_{j+1} = \varpi'$ are as in 10.6(d);
- $\varpi_j = \varpi, \varpi_{j+1} = \varpi'$ are as in 10.6(e).

Using 10.2(1) or 10.6(c) or 10.6(d) or 10.6(e) we see that for $j \in \{0, 1, \dots, k-1\}$ we have $I_{\varpi_j} \cong I_{\varpi_{j+1}}$ and $h(\varpi_j) = h(\varpi_{j+1})$. This proves (a).

The set of equivalence classes on $\mathring{\mathbf{E}}$ for \sim is denoted by $\underline{\mathring{\mathbf{E}}}$; it is a finite set.

10.8. For ϖ, ϖ' in \mathbf{E} we set

$$\begin{aligned} \tau(\varpi, \varpi') &= \sum_{(\alpha, n) \in \mathcal{R}_{\delta}^*; (\langle \varpi : \alpha \rangle + n - 2)(\langle \varpi' : \alpha \rangle + n - 2) < 0} \dim \mathfrak{g}_{\delta}^{\alpha, n}, \\ (a) \quad &- \sum_{(\alpha, n) \in \mathcal{R}_{\underline{0}}^*; (\langle \varpi : \alpha \rangle + n)(\langle \varpi' : \alpha \rangle + n) < 0} \dim \mathfrak{g}_{\underline{0}}^{\alpha, n}. \end{aligned}$$

Using 10.2(i) we see that when ϖ, ϖ' are in \mathbf{E}' , then

$$(b) \quad \tau(\varpi, \varpi') = -\dim \frac{\epsilon \mathbf{u}_{\underline{0}}^{\varpi} + \epsilon \mathbf{u}_{\underline{0}}^{\varpi'}}{\epsilon \mathbf{u}_{\underline{0}}^{\varpi} \cap \epsilon \mathbf{u}_{\underline{0}}^{\varpi'}} + \dim \frac{\epsilon \mathbf{u}_{\eta}^{\varpi} + \epsilon \mathbf{u}_{\eta}^{\varpi'}}{\epsilon \mathbf{u}_{\eta}^{\varpi} \cap \epsilon \mathbf{u}_{\eta}^{\varpi'}}.$$

10.9. We define G_{ϕ}, M_{ϕ} as in [LY, 3.6]. We show that:

(a) The obvious map $M_{\phi}/M_{\phi}^0 \rightarrow (G_{\underline{0}} \cap G_{\phi})/(G_{\underline{0}} \cap G_{\phi})^0$ is an isomorphism.

Recall that $\phi = (e, h, f)$ with $e \in \mathring{\mathfrak{m}}_{\eta}$. Let U (resp. U') be the unipotent radical of $G(e)^0$ (resp. $(M \cap G(e))^0$). We have $G(e) = G_{\phi}U$ (semidirect product). Taking fixed point sets of ϑ we obtain $G_{\underline{0}} \cap G(e) = (G_{\underline{0}} \cap G_{\phi})(G_{\underline{0}} \cap U)$ (semidirect product). We have $M \cap G(e) = M_{\phi}U'$ (semidirect product). Taking fixed point set of $\iota(t)$ for all $t \in \mathbf{k}^*$ we obtain $M_0 \cap G(e) = (M_0 \cap M_{\phi})(M_0 \cap U') = M_{\phi}(M_0 \cap U')$ (semidirect product). (We have used that $M_{\phi} \subset M_0$.) It follows that we have canonically

$$(M_0 \cap G(e))/(M_0 \cap G(e))^0 = M_{\phi}/M_{\phi}^0,$$

$$(G_{\underline{0}} \cap G(e))/(G_{\underline{0}} \cap G(e))^0 = (G_{\underline{0}} \cap G_{\phi})/(G_{\underline{0}} \cap G_{\phi})^0.$$

It remains to use that

$$(M_0 \cap G(e))/(M_0 \cap G(e))^0 = (G_{\underline{0}} \cap G(e))/(G_{\underline{0}} \cap G(e))^0;$$

see [LY, 3.8(a)].

Recall from [LY, 3.6] that Z is a maximal torus of $(G_{\phi} \cap G_{\underline{0}})^0$. Let H be the normalizer of Z in $(G_{\phi} \cap G_{\underline{0}})^0$. Let $H' = \{g \in G_{\underline{0}}; \text{Ad}(g)M = M, \text{Ad}(g)\mathfrak{m}_k = \mathfrak{m}_k \ \forall k \in \mathbf{Z}\}$. Note that M_0 is a normal subgroup of H' . Hence the groups H/Z , H'/M_0 are defined. We show:

(b) $H \subset H'$.

Let $g \in (G_{\phi} \cap G_{\underline{0}})^0$ be such that $gZg^{-1} = Z$. Let

$$(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*) = (gMg^{-1}, gM_0g^{-1}, \text{Ad}(g)\mathfrak{m}, \text{Ad}(g)\mathfrak{m}_*).$$

Note that $\mathcal{Z}_{M'}^0 = \mathcal{Z}_M^0 = Z$. Repeating the argument in the last paragraph in the proof of [LY, 3.6(a)], we see that

$$(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*).$$

(The argument is applicable since $g \in G_{\phi} \cap G_{\underline{0}}$.) We see that $g \in H'$; this proves (b).

We show:

(c) $H \cap M_0 = Z$.

From the injectivity of the map in (a) we see that $M_{\phi} \cap (G_{\underline{0}} \cap G_{\phi})^0 = M_{\phi}^0$. Hence we have

$$M_0 \cap (G_{\underline{0}} \cap G_{\phi})^0 \subset (M_0 \cap G_{\phi}) \cap (G_{\underline{0}} \cap G_{\phi})^0 \subset M_{\phi} \cap (G_{\underline{0}} \cap G_{\phi})^0 = M_{\phi}^0,$$

so that

$$M_0 \cap (G_{\underline{0}} \cap G_{\phi})^0 \subset M_{\phi}^0.$$

The opposite inclusion is also true since $M_{\phi} \subset M_0$ and $M_{\phi}^0 \subset G_{\underline{0}} \cap G_{\phi}$. It follows that

$$M_0 \cap (G_{\underline{0}} \cap G_{\phi})^0 = M_{\phi}^0 = Z.$$

The last equality is because e is distinguished in \mathfrak{m} . Now $H \cap M_0$ is the normalizer of Z in $M_0 \cap (G_{\phi} \cap G_{\underline{0}})^0$, that is, the normalizer of Z in Z . We see that $H \cap M_0 = Z$. This proves (c).

We show:

(d) $H' = M_0H$.

Since $H \subset H'$, $M_0 \subset H'$, we have $M_0H \subset H'$. Now let $g \in H'$. We show that $g \in M_0H$. Let $\phi' = (\text{Ad}(g)e, \text{Ad}(g)h, \text{Ad}(g)f)$. We have

$$\text{Ad}(g)e \in \overset{\circ}{\mathfrak{m}}_{\eta}, \text{Ad}(g)h \in \mathfrak{m}_0, \text{Ad}(g)f \in \mathfrak{m}_{-\eta}.$$

Since both $\text{Ad}(g)e, e$ are in $\overset{\circ}{\mathfrak{m}}_{\eta}$, we can find $g_1 \in M_0$ such that $\text{Ad}(g_1)\text{Ad}(g)e = e$. Replacing g by g_1g we can assume that we have $\text{Ad}(g)e = e$. Using [L4, 3.3] for J^M , we can find $g_2 \in M_0$ such that

$$(\text{Ad}(g_2)\text{Ad}(g)e, \text{Ad}(g_2)\text{Ad}(g)h, \text{Ad}(g_2)\text{Ad}(g)f) = (e, h, f).$$

We have $g_2g \in G_{\phi}$. Replacing g by g_2g we can assume that $g \in G_{\underline{0}} \cap G_{\phi}$. Using the surjectivity of the map in (a) we see that:

$$G_{\underline{0}} \cap G_{\phi} \subset M_{\phi}(G_{\underline{0}} \cap G_{\phi})^0.$$

Thus we can write g in the form $g_3 g'$ with $g_3 \in M_\phi$, $g' \in (G_0 \cap G_\phi)^0$. Replacing g by $g_3^{-1} g$ we see that we can assume that $g \in (G_0 \cap G_\phi)^0$. Since $\text{Ad}(g)M = M$, we see that $\text{Ad}(g)Z = Z$. Thus $g \in H$. This proves (d).

From (b),(c),(d) we see that:

(e) *The inclusion $H \subset H'$ induces an isomorphism $H/Z \xrightarrow{\sim} H'/M_0$. In particular, M_0 is the identity component of H' .*

10.10. Let $g \in H'$ (notation of 10.9). Then $\text{Ad}(g)$ restricts to an isomorphism $\mathfrak{m}_\eta \xrightarrow{\sim} \mathfrak{m}_\eta$. Let $\tilde{C}' = \text{Ad}(g)^* \tilde{C}$, a simple perverse sheaf in $\mathcal{Q}(\mathfrak{m}_\eta)$. We show:

(a) $\tilde{C}' \cong \tilde{C}$.

Using 10.9(e) we can assume that $g \in H$ (notation of 10.9). Since \tilde{C}, \tilde{C}' are intersection cohomology complexes attached to M_0 -equivariant irreducible local systems on \mathfrak{m}_η , they correspond to irreducible representations of

$$(M_0 \cap G(e))/(M_0 \cap G(e))^0 = M_\phi/M_\phi^0.$$

Hence it is enough to show that $\text{Ad}(g)$ induces the identity automorphism of M_ϕ/M_ϕ^0 . Using 10.9(a), we see that it is enough to show that $\text{Ad}(g)$ induces the identity automorphism of $(G_0 \cap G_\phi)/(G_0 \cap G_\phi)^0$. This is obvious since $g \in (G_0 \cap G_\phi)^0$. This proves (a).

10.11. Let $\varpi, \varpi' \in \mathbf{E}'$. Recall that $P_0 = e^{\epsilon \mathfrak{p}_0^{\varpi}} \subset G_0$, $P'_0 = e^{\epsilon \mathfrak{p}_0^{\varpi'}} \subset G_0$ are parabolic subgroups of G_0 with a common Levi subgroup \bar{M}_0 . Let $U_0 = U_{P_0}$, $U'_0 = U_{P'_0}$. Let X be the set of all $g \in G_0$ such that $\text{Ad}(g)\mathfrak{p}_*^{\varpi}$ and $\mathfrak{p}_*^{\varpi'}$ have a common splitting (as in 6.3). Note that X is a union of (P'_0, P_0) -double cosets in G_0 . We show:

(a) *We have $H' \subset X$ (notation of 10.9). Let $j : H'/M_0 \rightarrow P'_0 \backslash X/P_0$ be the map induced by the inclusion $H' \rightarrow X$. Then j is a bijection.*

If $g \in H'$, then $g \in G_0$ and $\text{Ad}(g)\mathfrak{m}_* = \mathfrak{m}_*$ hence \mathfrak{m}_* is a common splitting of $\text{Ad}(g)\mathfrak{p}_*^{\varpi}$ and $\mathfrak{p}_*^{\varpi'}$. Thus we have $g \in X$, so that the inclusion $H' \subset X$ holds. Let $g \in X$. Then $g \in G_0$ and $\text{Ad}(g)\mathfrak{p}_*^{\varpi}, \mathfrak{p}_*^{\varpi'}$ have a common splitting \mathfrak{m}'_* . Then $\text{Ad}(g)\mathfrak{m}_*, \mathfrak{m}'_*$ are splittings of $\text{Ad}(g)\mathfrak{p}_*^{\varpi}$ hence, by [LY, 2.7(a)], we have $\text{Ad}(gug^{-1})\text{Ad}(g)\mathfrak{m}_* = \mathfrak{m}'_*$ for some $u \in U_0$. Moreover, $\mathfrak{m}_*, \mathfrak{m}'_*$ are splittings of $\mathfrak{p}_*^{\varpi'}$ hence, by [LY, 2.7(a)], we have $\text{Ad}(u')\mathfrak{m}_* = \mathfrak{m}'_*$ for some $u' \in U'_0$. It follows that $\text{Ad}(gu)\mathfrak{m}_* = \text{Ad}(u')\mathfrak{m}_*$ hence $u'^{-1}gu \in H'$. Since $U_0 \subset P_0$, $U'_0 \subset P'_0$, we see that j is surjective. It remains to show that j is injective. Let g, g' be elements of H' such that $g' = p'_0 g p_0$ for some $p_0 \in P_0, p'_0 \in P'_0$. We must only show that $g' \in gM_0$. Let NM_0 be the normalizer of M_0 in G_0 . It is enough to show that the obvious map $NM_0/M_0 \rightarrow P'_0 \backslash G_0/P_0$ is injective. This is a well-known property of parabolic subgroups and their Levi subgroups in a connected reductive group. This completes the proof of (a).

Let $\mathcal{W} = H/Z$ (notation of 10.9). Let $w \in \mathcal{W}$ and let $g \in H$ be a representative of w . Now $\text{Ad}(g)$ restricts to an automorphism of Z which depends only on w ; this induces an isomorphism $Y_Z \xrightarrow{\sim} Y_Z$ and, by extension of scalars, a vector space isomorphism $\mathbf{E} \xrightarrow{\sim} \mathbf{E}$ denoted by $\varpi_1 \mapsto w\varpi_1$. For any $(\alpha, n, i) \in X_Z \times \mathbf{Z} \times \mathbf{Z}/m$, $\text{Ad}(g)$ defines an isomorphism $\mathfrak{g}_i^{\alpha, n} \xrightarrow{\sim} \mathfrak{g}_i^{w\alpha, n}$ where $w\alpha \in X_Z$ is given by $w\alpha(z) = \alpha(w^{-1}(z))$; hence for any i , $(\alpha, n) \mapsto (w\alpha, n)$ is a bijection $\mathcal{R}_i \xrightarrow{\sim} \mathcal{R}_i$. Moreover, for any $N \in \mathbf{Z}$ and any $(\alpha, n) \in \mathcal{R}_{\bar{N}}^*$, $w : \mathbf{E} \rightarrow \mathbf{E}$ restricts to a bijection from the affine hyperplane $\mathfrak{H}_{\alpha, n, N}$ to the affine hyperplane $\mathfrak{H}_{w\alpha, n, N}$. It follows that $w : \mathbf{E} \rightarrow \mathbf{E}$ restricts to a bijection $\mathbf{E}' \xrightarrow{\sim} \mathbf{E}'$.

We show:

(b) For any $\varpi \in \mathbf{E}'$ we have $\mathrm{Ad}(g)({}^\epsilon \mathfrak{p}_*^{\varpi}) = {}^\epsilon \mathfrak{p}_*^{w\varpi}$.

From 10.2(h) we have

$$\begin{aligned} \mathrm{Ad}(g)({}^\epsilon \mathfrak{p}_N^{\varpi}) &= \oplus_{(\alpha, n) \in \mathcal{R}_N; \langle \varpi : \alpha \rangle \geq 2N/\eta - n} \mathrm{Ad}(g) \mathfrak{g}_N^{\alpha, n} \\ &= \oplus_{(\alpha, n) \in \mathcal{R}_N; \langle \varpi : \alpha \rangle \geq 2N/\eta - n} (\mathfrak{g}_N^{w\alpha, n}), \\ {}^\epsilon \mathfrak{p}_N^{w\varpi} &= \oplus_{(\alpha, n) \in \mathcal{R}_N; \langle w\varpi : \alpha \rangle \geq 2N/\eta - n} (\mathfrak{g}_N^{\alpha, n}) \\ &= \oplus_{(\alpha', n) \in \mathcal{R}_N; \langle w\varpi : {}^w\alpha' \rangle \geq 2N/\eta - n} (\mathfrak{g}_N^{w\alpha', n}). \end{aligned}$$

It remains to use that $\langle \varpi : \alpha \rangle = \langle w\varpi : {}^w\alpha \rangle$.

10.12. For ϖ_1, ϖ_2 in \mathbf{E} we set

$$(a) \quad [\varpi_1 | \varpi_2] = (1 - v^2)^{-\dim Z} \sum_{w \in \mathcal{W}} v^{\tau(\varpi_2, w\varpi_1)} \in \mathbf{Q}(v).$$

(Here v is an indeterminate and $\tau(\varpi_2, w\varpi_1) \in \mathbf{Z}$ is as in 10.8.) When ϖ_1, ϖ_2 are in \mathbf{E}' we have:

$$(b) \quad \sum_{j \in \mathbf{Z}} d_j(\mathfrak{g}_\delta; \tilde{I}_{\varpi_1}, D(\tilde{I}_{\varpi_2})) v^{-j} = [\varpi_1 | \varpi_2].$$

This can be deduced from [LY, 6.4] as follows. The set X in [LY, 6.3] is described in our case in 10.11(a) in terms of the group H'/M_0 which, in turn, is identified in 10.9(e) with $\mathcal{W} = H/Z$; the integers $\tau(g)$ in 6.3 are identified with the integers $\tau(\varpi_2, w\varpi_1) \in \mathbf{Z}$ by 10.11(b). Finally, the set X' in [LY, 6.4] coincides with X in [LY, 6.4] by 10.10(a).

From the definitions we have $\tau(\varpi_2, w\varpi_1) = \tau(w^{-1}\varpi_2, \varpi_1) = \tau(\varpi_1, w^{-1}\varpi_2)$ for any $w \in \mathcal{W}$. It follows that

$$(c) \quad [\varpi_1 | \varpi_2] = [\varpi_2 | \varpi_1].$$

Let \mathbf{c}_1 (resp. \mathbf{c}_2) be the equivalence class for \sim in $\mathring{\mathbf{E}}$ that contains ϖ_1 (resp. ϖ_2). Using 10.7(a) we see that the right hand side of (b) depends only on $\mathbf{c}_1, \mathbf{c}_2$ and not on the specific elements $\varpi_1 \in \mathbf{c}_1 \cap \mathbf{E}', \varpi_2 \in \mathbf{c}_2 \cap \mathbf{E}'$. Hence for $\mathbf{c}_1, \mathbf{c}_2$ in $\mathring{\mathbf{E}}$ we can set $[\mathbf{c}_1 | \mathbf{c}_2] = [\varpi_1 | \varpi_2] \in \mathbf{Q}(v)$ for any $\varpi_1 \in \mathbf{c}_1 \cap \mathbf{E}', \varpi_2 \in \mathbf{c}_2 \cap \mathbf{E}'$.

10.13. Let \mathfrak{B} (resp. ${}^\xi \mathfrak{B}$) be the set of (isomorphism classes of) simple perverse sheaves in $\mathcal{Q}(\mathfrak{g}_\delta)$ (resp. ${}^\xi \mathcal{Q}(\mathfrak{g}_\delta)$). For any G_0 -orbit \mathcal{O} in $\mathfrak{g}_\delta^{nil}$ let $\mathfrak{B}_\mathcal{O}$ be the set of all $B \in \mathfrak{B}$ such that the support of B is equal to the closure of \mathcal{O} . We have $\mathfrak{B} = \sqcup_{\mathcal{O}} \mathfrak{B}_\mathcal{O}$. We define a map $\kappa : \mathfrak{B} \rightarrow \mathbf{N}$ by $\kappa(B) = \dim \mathcal{O}$ where $B \in \mathfrak{B}_\mathcal{O}$.

10.14. Let \mathbf{V}' be the $\mathbf{Q}(v)$ -vector space with basis $\{\tilde{T}_{\mathbf{c}}; \mathbf{c} \in \mathring{\mathbf{E}}\}$. On \mathbf{V}' we have a unique pairing $(:) : \mathbf{V}' \times \mathbf{V}' \rightarrow \mathbf{Q}(v)$ which is $\mathbf{Q}(v)$ -linear in the first argument, $\mathbf{Q}(v)$ -antilinear in the second argument (for $f \mapsto \bar{f}$) and such that for $\mathbf{c}_1, \mathbf{c}_2$ in $\mathring{\mathbf{E}}$ we have $(\tilde{T}_{\mathbf{c}_1} : \tilde{T}_{\mathbf{c}_2}) = [\mathbf{c}_1 | \mathbf{c}_2]$ (see 10.12).

Setting

$$\begin{aligned} \mathfrak{R}_l &= \{x \in \mathbf{V}'; (x : x') = 0 \quad \forall x' \in \mathbf{V}'\}, \\ \mathfrak{R}_r &= \{x \in \mathbf{V}'; (x' : x) = 0 \quad \forall x' \in \mathbf{V}'\}, \end{aligned}$$

we state the following.

Lemma 10.15. *We have $\mathfrak{R}_l = \mathfrak{R}_r$.*

The proof is given in 10.17.

10.16. We define a \mathbf{Q} -linear map

$$\tilde{\gamma} : \mathbf{V}' \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$$

by $\tilde{T}_{\mathbf{c}} \mapsto \tilde{I}_{\varpi}$, where ϖ is an element of $\mathbf{c} \cap \mathbf{E}'$. Now $\tilde{\gamma}$ is well-defined by 10.7(a) and is surjective by [LY, 8.4(b)], 10.4(b). We define a pairing

$$(\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_{\delta})) \times (\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_{\delta})) \rightarrow \mathbf{Q}((v))$$

(denoted by $(:)$) by requiring that it is $\mathbf{Q}(v)$ -linear in the first argument, $\mathbf{Q}(v)$ -antilinear in the second argument (for $f \mapsto \bar{f}$) and that its restriction

$$\mathcal{K}(\mathfrak{g}_{\delta}) \times \mathcal{K}(\mathfrak{g}_{\delta}) \rightarrow \mathbf{Q}((v))$$

is the same as the restriction of the pairing in [LY, 4.4(c)]. This restricts to a pairing

$$(\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})) \times (\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})) \rightarrow \mathbf{Q}((v))$$

(denoted again by $(:)$). We show that for b, b' in \mathbf{V}' we have

$$(a) \quad (\tilde{\gamma}(b) : \tilde{\gamma}(b')) = (b : b')$$

We can assume that $b = \tilde{T}_{\mathbf{c}}, b' = \tilde{T}_{\mathbf{c}'}$ with \mathbf{c}, \mathbf{c}' in $\overset{\circ}{\mathbf{E}}$. We must show that $\{\tilde{I}_{\varpi}, D(\tilde{I}_{\varpi'})\} = [\varpi | \varpi']$ where $\varpi \in \mathbf{c} \cap \mathbf{E}', \varpi' \in \mathbf{c}' \cap \mathbf{E}'$. This follows from 10.12(b). This proves (a).

10.17. Let

$$\begin{aligned} {}'\mathfrak{R}_l &= \{z \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta}); (z : z') = 0 \quad \forall z' \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})\}, \\ {}'\mathfrak{R}_r &= \{z \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta}); (z' : z) = 0 \quad \forall z' \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})\}. \end{aligned}$$

We show:

$$(a) \quad {}'\mathfrak{R}_l = {}'\mathfrak{R}_r = 0.$$

Now $\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ has a $\mathbf{Q}(v)$ -basis formed by ${}^{\xi}\mathfrak{B} = \{B_1, B_2, \dots, B_r\}$. From [LY, 0.12] we see that $(B_j : B_{j'}) \in \delta_{j,j'} + v\mathbf{N}[[v]]$ for j, j' in $[1, r]$.

Assume that $\beta = \sum_{j \in [1, r]} f_j B_j \in {}'\mathfrak{R}_l$, where $f_j \in \mathbf{Q}(v)$ are not all zero. We must show that this is a contradiction. We can assume that $f_j \in \mathbf{Q}[v]$ for all j and $f_{j_0} - c_0 \in v\mathbf{Q}[v]$ for some $j_0 \in [1, r]$ and some $c_0 \in \mathbf{Q} - \{0\}$. Then $0 = (\beta : B_{j_0}) \in c_0 + v\mathbf{Q}[[v]]$, a contradiction. This proves that ${}'\mathfrak{R}_l = 0$.

Next we assume that $\beta = \sum_{j \in [1, r]} f_j B_j \in {}'\mathfrak{R}_r$, where $f_j \in \mathbf{Q}(v)$ are not all zero. We must show that this is a contradiction. We can assume that $\bar{f}_j \in \mathbf{Q}[v]$ for all j and $\bar{f}_{j_0} - c_0 \in v\mathbf{Q}[v]$ for some $j_0 \in [1, r]$ and some $c_0 \in \mathbf{Q} - \{0\}$. Then $0 = (B_{j_0} : \beta) \in c_0 + v\mathbf{Q}[[v]]$, a contradiction. This proves that ${}'\mathfrak{R}_r = 0$. This proves (a).

We show:

$$(b) \quad \mathfrak{R}_l = \tilde{\gamma}^{-1}({}'\mathfrak{R}_l).$$

Let $x \in \mathfrak{R}_l$. From 10.16(a) we see that $(\tilde{\gamma}(x) : \tilde{\gamma}(x')) = 0$ for any $x' \in \mathbf{V}'$. Since $\tilde{\gamma}$ is surjective, it follows that $(\tilde{\gamma}(x) : z') = 0$ for any $z' \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$. Thus, $\tilde{\gamma}(x) \in {}'\mathfrak{R}_l$. Conversely, assume that $x \in \mathbf{V}'$ and $\tilde{\gamma}(x) \in {}'\mathfrak{R}_l$. From 10.16(a) we see that for any $x' \in \mathbf{V}'$ we have $(x : x') = (\tilde{\gamma}(x) : \tilde{\gamma}(x')) = 0$. Thus $x \in \mathfrak{R}_l$. This proves (b).

An entirely similar proof shows that:

$$(c) \mathfrak{R}_r = \tilde{\gamma}^{-1}({}'\mathfrak{R}_r).$$

From (a),(b),(c) we see that $\mathfrak{R}_l = \mathfrak{R}_r = \tilde{\gamma}^{-1}(0)$. This proves Lemma 10.15.

10.18. Definition of \mathbf{V} . We define $\mathbf{V} = \mathbf{V}'/\mathfrak{R}_l = \mathbf{V}'/\mathfrak{R}_r$ (see Lemma 10.15). Note that $(:)$ on \mathbf{V}' induces a pairing $\mathbf{V} \times \mathbf{V} \rightarrow \mathbf{Q}(v)$ (denoted again by $(:)$) which is $\mathbf{Q}(v)$ -linear in the first argument, $\mathbf{Q}(v)$ -antilinear in the second argument (for $f \mapsto \bar{f}$).

10.19. From the proof of Lemma 10.15 we see that $\tilde{\gamma}$ induces a $\mathbf{Q}(v)$ -linear isomorphism

$$(a) \quad \gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$$

and that for b, b' in \mathbf{V} we have

$$(b) \quad (\gamma(b) : \gamma(b')) = (b : b').$$

From (a) we deduce the following result.

Proposition 10.20. *The number of simple perverse sheaves (up to isomorphism) in ${}^{\xi}\mathcal{Q}(\mathfrak{g}_{\delta})$ is equal to $\dim_{\mathbf{Q}(v)} \mathbf{V}$.*

10.21. We define a \mathbf{Q} -linear involution $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$ by $\overline{f\tilde{T}_{\mathbf{c}}} = \tilde{f}\tilde{T}_{\mathbf{c}}$ for any $f \in \mathbf{Q}(v)$, $\mathbf{c} \in \mathring{\mathbf{E}}$; here \tilde{f} is as in [LY, 0.12]. We show:

(a) *For any x, x' in \mathbf{V}' we have $(x : x') = (\bar{x}' : \bar{x})$.*

We can assume that $x = \tilde{T}_{\mathbf{c}}, x' = \tilde{T}_{\mathbf{c}'}$ for some \mathbf{c}, \mathbf{c}' in $\mathring{\mathbf{E}}$. We must show that

$$[\mathbf{c}|\mathbf{c}'] = [\mathbf{c}'|\mathbf{c}],$$

which follows directly from 10.12(c). This proves (a).

We show:

(b) $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$ preserves $\mathfrak{R}_l = \mathfrak{R}_r$ (see Lemma 10.15) hence it induces an $\mathbf{Q}(v)$ -semilinear involution $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$ (with respect to $f \mapsto \bar{f}$).

Assume that $x \in \mathfrak{R}_l$. We have $(x : x') = 0$ for all $x' \in \mathbf{V}$. Hence, by (a), we have $(\bar{x}' : \bar{x}) = 0$ for all $x' \in \mathbf{V}$. Since $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$ is surjective, it follows that $\bar{x} \in \mathfrak{R}_r$. This proves (b).

10.22. Now let $\eta_1 \in \mathbf{Z} - \{0\}$ be such that $\underline{\eta}_1 = \delta$. Recall that in 10.1 we have fixed $\xi \in \underline{\mathfrak{T}}_{\eta}$ and a representative $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_{\eta}$ for ξ . Let $\dot{\xi}_1 = (M, M_0, \mathfrak{m}, \mathfrak{m}_{(*)}, \tilde{C}) \in \mathfrak{T}_{\eta_1}$ be as in [LY, 3.9] and let $\xi_1 \in \underline{\mathfrak{T}}_{\eta_1}$ be the G_0 -orbit of $\dot{\xi}_1$. Note that the \mathbf{Q} -vector space \mathbf{E} defined as in 10.1 in terms of $\dot{\xi}$ is the same as \mathbf{E} defined in terms of $\dot{\xi}_1$. Moreover, the subset $\mathring{\mathbf{E}}$, the equivalence relation \sim on it, and the set $\mathring{\mathbf{E}}$ defined as in 10.7 in terms of $\dot{\xi}$ are the same as the analogous objects defined in terms of $\dot{\xi}_1$. Also, the pairings $\tau : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{Z}$ and $[?|?] : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{Q}(v)$ defined in 10.8 and 10.12 in terms of $\dot{\xi}$ are the same as those defined in terms of $\dot{\xi}_1$.

The subset \mathbf{E}' of \mathbf{E} defined as in 10.1 in terms of $\dot{\xi}$ is not in general the same as the analogous subset \mathbf{E}'_1 of \mathbf{E} defined in terms of $\dot{\xi}_1$. However, for $\mathbf{c}_1, \mathbf{c}_2$ in $\mathring{\mathbf{E}}$, the quantity $[\mathbf{c}_1|\mathbf{c}_2] \in \mathbf{Q}(v)$ defined in 10.12 in terms of $\dot{\xi}$ is the same as that defined in terms of $\dot{\xi}_1$. (It is equal to $[\varpi_1|\varpi_2]$ for any $\varpi_1 \in \mathbf{c}_1 \cap \mathbf{E}' \cap \mathbf{E}'_1$, $\varpi_2 \in \mathbf{c}_2 \cap \mathbf{E}' \cap \mathbf{E}'_1$. Hence the vector space \mathbf{V}' , the pairing $(:)$ on it and its quotient \mathbf{V} defined in 10.14

and 10.18 in terms of $\dot{\xi}$ are the same as those defined in terms of $\dot{\xi}_1$. The involutions $\bar{\cdot}: \mathbf{V}' \rightarrow \mathbf{V}'$, $\bar{\cdot}: \mathbf{V} \rightarrow \mathbf{V}$ defined in 10.21 in terms of $\dot{\xi}$ are the same as those defined in terms of $\dot{\xi}_1$.

11. THE \mathcal{A} -LATTICE $\mathbf{V}_{\mathcal{A}}$

In this section we give a combinatorial definition of an \mathcal{A} -lattice $\mathbf{V}_{\mathcal{A}}$ in the $\mathbf{Q}(v)$ -vector space \mathbf{V} and a signed basis \mathbf{B}' of it. It turns out that the $\mathbf{Z}/2$ -orbits on \mathbf{B}' for the $\mathbf{Z}/2$ -action $b \mapsto -b$ are in natural bijection with the simple perverse sheaves in the block ${}^{\xi}\mathcal{Q}(\mathfrak{g}_{\delta})$.

11.1. In this section (except in 11.5, 11.6, 11.13) we preserve the setup and notation of 10.1, 10.2 and assume that $\varpi \in \mathbf{E}''$ (see 10.3). Let

$$\mathfrak{g}^{\phi} = \{y \in \mathfrak{g}; [y, e] = 0, [y, h] = 0, [y, f] = 0\}.$$

Let \mathfrak{z} be the center of \mathfrak{m} . Note that $\mathfrak{m} \cap \mathfrak{g}^{\phi} = \mathfrak{z}$ (since e is distinguished in \mathfrak{m}) and \mathfrak{z} is a Cartan subalgebra of the reductive Lie algebra \mathfrak{g}^{ϕ} . (This has already been proved at level of groups in [LY, 3.6].) For any $\alpha \in X_Z$ let

$$(\mathfrak{g}_{\underline{0}}^{\phi})^{\alpha} = \mathfrak{g}_{\underline{0}}^{\alpha} \cap \mathfrak{g}^{\phi}.$$

Let

$$\mathfrak{g}_{\underline{0}, \varpi}^{\phi} = \bigoplus_{\alpha \in X_Z; \langle \varpi, \alpha \rangle = 0} (\mathfrak{g}_{\underline{0}}^{\phi})^{\alpha}.$$

This is a Levi subalgebra (containing \mathfrak{z}) of a parabolic subalgebra of $\mathfrak{g}_{\underline{0}}^{\phi}$. Let \mathcal{B} be the variety of Borel subalgebras of $\mathfrak{g}_{\underline{0}, \varpi}^{\phi}$, let $d(\varpi) = \dim \mathcal{B}$ and let

$$a_{\varpi} = \sum_j v^{-2s_j} v^{d(\varpi)} \in \mathcal{A},$$

where $\rho_{\mathcal{B}}! \bar{\mathbf{Q}}_l = \bigoplus_j \bar{\mathbf{Q}}_l[-2s_j]$.

Erratum to [L4]. On page 202, line 1 of 16.8, replace “algebraic group M ” by “algebraic group M with a given Lie algebra homomorphism ϕ from \mathfrak{s} to the Lie algebra of M ”.

On page 202, line 4 of 16.8, replace “Borel subgroups of M ” by “Borel subgroups of the connected centralizer of $\phi(\mathfrak{s})$ in M ”.

11.2. The subset

$$(a) \quad \{a_{\varpi}^{-1} \tilde{T}_{\mathbf{c}} \in \mathbf{V}'; \mathbf{c} \in \underline{\mathbf{E}}, \varpi \in \mathbf{E}'' \cap \mathbf{c}\}$$

of \mathbf{V}' (see 10.14) is finite. Indeed, when \mathbf{c} is fixed, the subgroup $\mathfrak{g}_{\underline{0}, \varpi}^{\phi}$ of $\mathfrak{g}_{\underline{0}}^{\phi}$ takes only finitely many values for ϖ in $\mathbf{E}'' \cap \mathbf{c}$ hence a_{ϖ} takes only finitely many values.

Let $\mathbf{V}'_{\mathcal{A}}$ be the \mathcal{A} -submodule of \mathbf{V}' generated by (a). Let $\mathbf{V}_{\mathcal{A}}$ be the image of $\mathbf{V}'_{\mathcal{A}}$ under the obvious linear map $\mathbf{V}' \rightarrow \mathbf{V}$. The following result will be proved in 11.8.

(b) $\mathbf{V}_{\mathcal{A}}$ is a free \mathcal{A} -module such that the obvious $\mathbf{Q}(v)$ -linear map $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathbf{V}_{\mathcal{A}} \rightarrow \mathbf{V}$ is an isomorphism.

11.3. Let $\mathfrak{h} = \tilde{\mathfrak{l}}^{\varpi}$, $\mathfrak{h}^{\phi} = \mathfrak{h} \cap \mathfrak{g}^{\phi}$. We show:

$$(a) \quad \mathfrak{g}_{0,\varpi}^{\phi} = \mathfrak{h}^{\phi}.$$

From 10.3(b) we have

$$\mathfrak{h} = \bigoplus_{N \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_{\underline{N}}; n=2N/\eta, \langle \varpi: \alpha \rangle = 0} (\mathfrak{g}_{\underline{N}}^{\alpha, n}).$$

Using this and the definitions we have

$$\begin{aligned} \{y \in \mathfrak{h}; [y, h] = 0\} &= \bigoplus_{N \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_{\underline{N}}; n=2N/\eta=0, \langle \varpi: \alpha \rangle = 0} (\mathfrak{g}_{\underline{N}}^{\alpha, n}) \\ &= \bigoplus_{(\alpha, n) \in \mathcal{R}_{\underline{0}}; n=0, \langle \varpi: \alpha \rangle = 0} (\mathfrak{g}_{\underline{0}}^{\alpha, n}) \end{aligned}$$

and (a) follows.

11.4. Recall that $\varpi \in \mathbf{E}''$. Let $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}_*^{\varpi}$, $\mathfrak{u}_* = {}^{\epsilon}\mathfrak{u}_*^{\varpi}$. We can find $\lambda' \in Y_Z$ such that $\langle \lambda' : \alpha \rangle \neq 0$ for any i and any $(\alpha, n) \in \mathcal{R}_i^*$. Let $\varpi' = \varpi + \frac{1}{b}\lambda' \in \mathbf{E}$. Let $\mathfrak{p}'_* = {}^{\epsilon}\mathfrak{p}'_*^{\varpi'}$, $\mathfrak{u}'_* = {}^{\epsilon}\mathfrak{u}'_*^{\varpi'}$ and $\mathfrak{m}'_* = {}^{\epsilon}\tilde{\mathfrak{m}}_*^{\varpi'}$. Assume that b is sufficiently large; then $\varpi' \in \mathbf{E}'$ hence by 10.2 we have $\mathfrak{m}'_* = \mathfrak{m}_*$. The same argument as in 10.6(a) shows that $\mathfrak{p}'_N \subset \mathfrak{p}_N$ for all N . Since b is large and $\varpi' = \varpi + \frac{1}{b}\lambda'$, we see that ϖ' , ϖ are very close, so that

$$(a) \quad \varpi' \sim \varpi.$$

For $N \in \mathbf{Z}$ let \mathfrak{q}_N be the image of \mathfrak{p}'_N under the obvious projection $\mathfrak{p}_N \rightarrow \mathfrak{h}_N$. From 10.5(a),(b), we see that $\mathfrak{q} = \bigoplus_N \mathfrak{q}_N$ is a parabolic subalgebra of \mathfrak{h} and \mathfrak{m} is a Levi subalgebra of \mathfrak{q} . Moreover, if u_N is the image of \mathfrak{u}'_N under \mathfrak{p}_N , then $u = \bigoplus_N u_N$ is the nilradical of \mathfrak{q} .

From 10.3(b) we see that the \mathbf{Z} -grading of \mathfrak{h} is η -rigid and that $e \in \mathring{\mathfrak{h}}_{\eta}$, so that $\mathring{\mathfrak{m}}_{\eta} \subset \mathring{\mathfrak{h}}_{\eta}$. Let $A_{\varpi} \in \mathcal{Q}(\mathfrak{h}_{\eta})$ be the simple perverse sheaf on \mathfrak{h}_{η} such that the support of A_{ϖ} is \mathfrak{h}_{η} and $A_{\varpi}|_{\mathring{\mathfrak{m}}_{\eta}}^{\circ}$ is equal up to shift to $\tilde{C}|_{\mathring{\mathfrak{m}}_{\eta}}$.

Applying 1.8(b) and the transitivity formula 4.2(a) we deduce

$$\begin{aligned} I_{\varpi'} &= {}^{\epsilon}\widetilde{\text{Ind}}_{\mathfrak{p}_{\eta}}^{\mathfrak{g}\delta}(\tilde{C}) = {}^{\epsilon}\widetilde{\text{Ind}}_{\mathfrak{p}_{\eta}}^{\mathfrak{g}\delta}(\text{ind}_{\mathfrak{q}_{\eta}}^{\mathfrak{h}_{\eta}}(\tilde{C}))[\dim \mathfrak{u}'_0 + \dim \mathfrak{u}'_{\eta} - \dim \mathfrak{u}_0 - \dim \mathfrak{u}_{\eta}] \\ (b) \quad &\cong \bigoplus_j {}^{\epsilon}\widetilde{\text{Ind}}_{\mathfrak{p}_{\eta}}^{\mathfrak{g}\delta}(A_{\varpi})[-2s_j][\dim \mathfrak{m}_{\eta} - \dim \mathfrak{h}_{\eta} + \dim \mathfrak{u}'_0 + \dim \mathfrak{u}'_{\eta} \\ &\quad - \dim \mathfrak{u}_0 - \dim \mathfrak{u}_{\eta}] \\ &= \bigoplus_j {}^{\epsilon}\widetilde{\text{Ind}}_{\mathfrak{p}_{\eta}}^{\mathfrak{g}\delta}(A_{\varpi})[-2s_j][d(\varpi)], \end{aligned}$$

where we have used the equality:

$$(c) \quad \dim \mathfrak{m}_{\eta} - \dim \mathfrak{h}_{\eta} + \dim \mathfrak{u}'_0 + \dim \mathfrak{u}'_{\eta} - \dim \mathfrak{u}_0 - \dim \mathfrak{u}_{\eta} = d(\varpi).$$

We now prove (c). Let u' be the nilradical of the parabolic subalgebra of \mathfrak{h} that contains \mathfrak{m} and is opposed to \mathfrak{q} . We have $u' = \bigoplus_N u'_N$ where $u'_N = u' \cap \mathfrak{h}_N$. Now u_0, u'_0 are nilradicals of two opposite parabolic subalgebras of \mathfrak{h}_0 hence $\dim u_0 = \dim u'_0$. We have $\dim \mathfrak{h}_{\eta} = \dim u_{\eta} + \dim u'_{\eta} + \dim \mathfrak{m}_{\eta}$, $\dim u'_0 - \dim u_0 = \dim u_0$, $\dim u'_{\eta} - \dim u_{\eta} = \dim u_{\eta}$. Hence the left-hand side of (c) equals $\dim u_0 - \dim u'_{\eta} = \dim u'_0 - \dim u'_{\eta}$. Now u' is normalized by \mathfrak{m} hence by the Lie subalgebra \mathfrak{s} of \mathfrak{m} spanned by e, h, f . Note that u'_0 (resp. u'_{η}) is the 0-(resp. 2-) eigenspace of $\text{ad}(h) : u' \rightarrow u'$. By the representation theory of \mathfrak{s} , the map $\text{ad}(e) : u'_0 \rightarrow u'_{\eta}$ is surjective and its kernel is exactly the space of \mathfrak{s} -invariants in u' that is $u' \cap \mathfrak{h}^{\phi}$. We see that $\dim u'_0 - \dim u'_{\eta} = \dim(u' \cap \mathfrak{h}^{\phi})$. Now $u' \cap \mathfrak{h}^{\phi}$ is the nilradical of a

parabolic subalgebra of \mathfrak{h}^ϕ with Levi subgroup $\mathfrak{m} \cap \mathfrak{h}^\phi = \mathfrak{z}$ (see 11.1) hence is the nilradical of a Borel subalgebra of \mathfrak{h}^ϕ (which equals $\mathfrak{g}_{0,\varpi}^\phi$ by 11.3(a)). By definition, the dimension of this nilradical is equal to $d(\varpi)$. This proves (c) and hence also (b).

Now (b) implies the following equality in ${}^\xi\mathcal{K}(\mathfrak{g}_\delta)$:

$$I_{\varpi'} = a_{\varpi}({}^\epsilon\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_{\varpi})).$$

Using this and (a) we see that for any $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$ and any $\varpi \in \mathbf{E}'' \cap \mathbf{c}$ we have

$$(d) \quad a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}}) = {}^\epsilon\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_{\varpi}) \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi\mathcal{K}(\mathfrak{g}_\delta).$$

Since ${}^\epsilon\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_{\varpi}) \in {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$, we see that:

(e) *For any $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$ and any $\varpi \in \mathbf{E}'' \cap \mathbf{c}$ we have $a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}}) \in {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$.*

The following result will be proved in 11.7.

(f) *The \mathcal{A} -module ${}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ is generated by the elements $a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}})$ for various $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$ and $\varpi \in \mathbf{E}'' \cap \mathbf{c}$.*

From (f) we deduce:

(g) *The isomorphism $\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ in Proposition 10.19(a) restricts to an isomorphism of \mathcal{A} -modules $\gamma_{\mathcal{A}} : \mathbf{V}_{\mathcal{A}} \xrightarrow{\sim} {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$.*

11.5. Let \mathfrak{p}_* be an ϵ -spiral with a splitting \mathfrak{h}_* and with nilradical \mathfrak{u}_* ; let $A \in \mathcal{Q}(\mathfrak{h}_\eta)$ be a simple perverse sheaf. We show:

(a) *Let \mathcal{X} be the collection of all $B \in \mathfrak{B}$ such that some shift of B is a direct summand of $\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$. Then the map $\mathcal{X} \rightarrow \mathfrak{T}_\eta$, $B \mapsto \psi(B)$ is constant.*

We can find a parabolic subalgebra \mathfrak{q} of \mathfrak{h} and a Levi subalgebra \mathfrak{m}' of \mathfrak{q} such that $\mathfrak{q} = \oplus_N \mathfrak{q}_N$, $\mathfrak{m}' = \oplus_N \mathfrak{m}'_N$ where $\mathfrak{q}_N = \mathfrak{q} \cap \mathfrak{h}_N$, $\mathfrak{m}'_N = \mathfrak{m}' \cap \mathfrak{h}_N$ and a cuspidal perverse sheaf C in $\mathcal{Q}(\mathfrak{m}_\eta)$ such that some shift of A is a direct summand of $\text{ind}_{\mathfrak{q}_\eta}^{\mathfrak{h}_\eta}(C)$. Let $M' = e^{\mathfrak{m}'}$, $M'_0 = e^{\mathfrak{m}'_0}$. Setting $\mathfrak{p}'_N = \mathfrak{u}_N + \mathfrak{q}_N$ for any $N \in \mathbf{Z}$, we see from [LY, 2.8(a)] that \mathfrak{p}'_* is an ϵ -spiral and from [LY, 2.8(b)] that \mathfrak{m}'_* is a splitting of \mathfrak{p}'_* . We see that $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C) \in \mathfrak{T}_\eta$. Let ξ' be the element of \mathfrak{T}_η determined by $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C)$. If $B \in \mathcal{X}$, then (by [LY, 4.2]) some shift of B is a direct summand of $\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ hence $\psi(B) = \xi'$. This proves (a).

We say that $\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ (as in (a)) has type $\xi' \in \mathfrak{T}_\eta$ if $\psi(B) = \xi'$ for any $B \in \mathcal{X}$.

11.6. Recall from 8.4(a) that the \mathcal{A} -module $\mathcal{K}(\mathfrak{g}_\delta)$ is generated by the classes of ϵ -quasi-monomial objects of $\mathcal{Q}(\mathfrak{g}_\delta)$. Using this and 11.5(a), we deduce that in the direct sum decomposition $\mathcal{K}(\mathfrak{g}_\delta) = \oplus_{\xi \in \mathfrak{T}_\eta} {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ (see [LY, 6.7]), any summand ${}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ is generated as an \mathcal{A} -module by the classes of η -quasi-monomial objects in $\mathcal{Q}(\mathfrak{g}_\delta)$ of type ξ .

11.7. We prove 11.4(f). (Thus we are again in the setup of 11.1.) Using 11.6, we see that it is enough to show that if A' is an η -quasi-monomial object in $\mathcal{Q}(\mathfrak{g}_\delta)$ of type ξ , then the class of A' in $\mathcal{K}(\mathfrak{g}_\delta)$ is of the form $a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}})$ for some $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$ and $\varpi \in \mathbf{E}'' \cap \mathbf{c}$. We can find:

(a) $\mathfrak{p}_*, \mathfrak{h}_*, A, \mathfrak{q}_*, \mathfrak{p}'_*, (M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C) \in \mathfrak{T}_\eta$ (representing ξ) as in 11.5 such that $A' = \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$; moreover, we can assume that the \mathbf{Z} -grading of $\mathfrak{h} = \oplus_N \mathfrak{h}_N$ is

η -rigid and $\overset{\circ}{\mathfrak{m}}'_\eta \subset \overset{\circ}{\mathfrak{h}}_\eta$. Replacing the data (a) by a G_0 -conjugate we can assume in addition that $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C)$ is equal to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ in 10.1.

Let $H = e^h$. Since \mathfrak{h}_* is η -rigid, there exists $\iota' \in Y_H$ such that:

- (i) $\iota'_k = \mathfrak{h}_{k\eta/2}$ if $k \in \mathbf{Z}$, $k\eta/2, \iota'_k = 0$ if $k \in \mathbf{Z}$, $k\eta/2 \notin \mathbf{Z}$ and
- (ii) $\iota' = \iota_{\phi'}$ for some $\phi' = (e', h', f') \in J^H$ such that $e' \in \overset{\circ}{\mathfrak{h}}_\eta$, $h' \in \mathfrak{h}_0$, $f' \in \mathfrak{h}_{-\eta}$.

Since $\overset{\circ}{\mathfrak{m}}_\eta \subset \overset{\circ}{\mathfrak{h}}_\eta$ and $e \in \overset{\circ}{\mathfrak{m}}_\eta$, we see that e, e' are in the same M_0 -orbit. Hence we can find $g \in M_0$ such that $\text{Ad}(g)$ conjugates e', f', h', ι' to e, f, h, ι . Applying $\text{Ad}(g)$ (which preserves \mathfrak{h}_k) to (i) and (ii) we see that we can assume that $\iota' = \iota$, $\phi' = \phi$.

Recall from 10.3 that $\tilde{\mathfrak{l}}_N^\phi = {}_{2N/\eta}\mathfrak{g}_N$ for $N \in \mathbf{Z}$ such that $2N/\eta \in \mathbf{Z}$. Using (i) with $\iota' = \iota$ we see that $\mathfrak{h}_N \subset \tilde{\mathfrak{l}}_N^\phi$ for any $N \in \mathbf{Z}$ such that $2N/\eta \in \mathbf{Z}$. Using 10.4(c),(d) we see that for some $\varpi \in \mathbf{E}''$ we have $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^{\varpi}$, $\mathfrak{h}_* = {}^\epsilon \tilde{\mathfrak{l}}_*^{\varpi}$. Using now 11.4(d), we see that $A' = a_{\varpi}^{-1} \tilde{\gamma}(\tilde{T}_{\mathbf{c}})$, where $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$ contains ϖ . This completes the proof of 11.4(f), hence that of 11.4(g).

11.8. We can now prove 11.2(b). Using 11.4(g), 11.2(b) is reduced to the following obvious statement: ${}^\xi \mathcal{K}(\mathfrak{g}_\delta)$ is a free \mathcal{A} -module.

11.9. We define a \mathbf{Q} -linear map

$$\bar{\cdot}: \mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_\delta) \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_\delta)$$

by $\overline{fB} = \bar{f}B$ for any $f \in \mathbf{Q}(v)$ and any $B \in \mathfrak{B}$ (see 10.13); here \bar{f} is as in 0.12. This restricts to a \mathbf{Q} -linear map

$$\bar{\cdot}: \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi \mathcal{K}(\mathfrak{g}_\delta) \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$$

and to a \mathbf{Z} -linear map ${}^\xi \mathcal{K}(\mathfrak{g}_\delta) \rightarrow {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$. We show:

- (a) $\overline{I_\varpi} = I_\varpi$ for any $\varpi \in \mathbf{E}'$.

In ${}^\xi \mathcal{Q}(\mathfrak{g}_\delta)$ we have $I_\varpi = \sum_{B \in {}^\epsilon \mathfrak{B}} f_B B$ where $f_B \in \mathcal{A}$. It is enough to prove that $\bar{f}_B = f_B$ for all B .

We set $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^{\varpi}$. Let σ be an automorphism of order 2 of $\bar{\mathbf{Q}}_l$ such that $\sigma(z) = z^{-1}$ for any root of 1 in $\bar{\mathbf{Q}}_l$. Applying σ to $K \in \mathcal{Q}(\mathfrak{m}_\eta)$ (resp. $K \in \mathcal{Q}(\mathfrak{g}_\delta)$) we obtain $K^\sigma \in \mathcal{Q}(\mathfrak{m}_\eta)$ (resp. $K^\sigma \in \mathcal{Q}(\mathfrak{g}_\delta)$). Note that $K \mapsto K^\sigma$ commutes with shifts; moreover, we have

$$(b) \quad (\widetilde{{}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C})})^\sigma = \widetilde{{}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}^\sigma)}.$$

Moreover, if K is a simple perverse sheaf in $\mathcal{Q}(\mathfrak{m}_\eta)$ or in $\mathcal{Q}(\mathfrak{g}_\delta)$, we have

$$(c) \quad K^\sigma \cong D(K),$$

since K restricted to an open dense subset of its support is a local system with finite monodromy. By (b),(c) and [LY, 4.1(d)] we have

$$D(\widetilde{{}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C})}) = \widetilde{{}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(\tilde{C}))} = \widetilde{{}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}^\sigma)} = (\widetilde{{}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C})})^\sigma,$$

hence

$$(D(I_\varpi))^\sigma = I_\varpi$$

and $\sum_B \bar{f}_B D(B^\sigma) = \sum_B f_B B$. Using this and (c) for $K = B$, we see that $\bar{f}_B = f_B$ for all B ; this proves (a).

11.10. We show:

(a) $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$ (see 10.21) restricts to an involution $\mathbf{V}'_{\mathcal{A}} \rightarrow \mathbf{V}'_{\mathcal{A}}$ and $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$ (see 10.21) restricts to an involution $\mathbf{V}_{\mathcal{A}} \rightarrow \mathbf{V}_{\mathcal{A}}$ (these restrictions are denoted again by $\bar{\cdot}$).

It is enough to note that for any $\varpi \in \mathbf{E}''$ we have $\overline{a_{\varpi}} = a_{\varpi}$.

We show:

(b) $\tilde{\gamma} : \mathbf{V}' \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ is compatible with the maps $\bar{\cdot}$ in the two sides.

This follows from 11.9(a).

We have the following result.

Proposition 11.11.

(a) Let \mathbf{B}' be the set of all $b \in \mathbf{V}_{\mathcal{A}}$ such that $\bar{b} = b$ and $(b : b) \in 1 + v\mathbf{Z}[[v]]$. Then \mathbf{B}' is a signed basis of the \mathcal{A} -module $\mathbf{V}_{\mathcal{A}}$ (that is, the union of a basis with (-1) times that basis).

(b) For $b \in \mathbf{B}'$ we have $(b : b) \in 1 + v\mathbf{N}[[v]]$.

(c) There is a unique \mathcal{A} -basis \mathbf{B} of $\mathbf{V}_{\mathcal{A}}$ such that $\mathbf{B} \subset \mathbf{B}'$ and for any $\mathbf{c} \in \underline{\mathbf{E}}$ and any $\varpi \in \mathbf{E}'' \cap \mathbf{c}$, the image of $a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}}$ in $\mathbf{V}_{\mathcal{A}}$ is an $\mathbf{N}[v, v^{-1}]$ -linear combination of elements in \mathbf{B} .

It is enough to prove the analogous statements where $\mathbf{V}_{\mathcal{A}}$ is identified via γ with ${}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ with (\cdot) as in 4.4(c) and with $\bar{\cdot}$ as in 11.9 (we use 10.19(b), 11.10(b)). Let ${}^{\xi}\mathfrak{B} = \{B_1, B_2, \dots, B_r\}$ (see 10.13). From 0.12 we have $(B_j : B_{j'}) \in \delta_{j,j'} + h_{j,j'}$, where $h_{j,j'} \in v\mathbf{N}[[v]]$ for all j, j' in $[1, r]$. From the definition (see 11.9) we have $\bar{B}_j = B_j$ for $j = 1, \dots, r$. Now let $b \in {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ be such that $\bar{b} = b$ and $(b : b) \in 1 + v\mathbf{Z}[[v]]$. To prove (a), it is enough to show that $b = \pm B_j$ for some j . We can write $b = \sum_{j=1}^r f_j B_j$, where $f_j \in \mathcal{A}$ satisfy $\bar{f}_j = f_j$ and $\sum_{j,j' \in [1,r]} \bar{f}_j f_{j'} (\delta_{j,j'} + h_{j,j'}) \in 1 + v\mathbf{Z}[[v]]$ hence $\sum_{j,j' \in [1,r]} f_j f_{j'} (\delta_{j,j'} + h_{j,j'}) \in 1 + v\mathbf{Z}[[v]]$. We can find $c \in \mathbf{Z}$ such that $f_j = f_{j,c} v^c \pmod{v^{c+1}\mathbf{Z}[v]}$ where $f_{j,c} \in \mathbf{Z}$ for all j and $f_{j,c} \neq 0$ for some j . We have $\sum_{j \in [1,r]} f_{j,c}^2 v^{2c} + v^{2c+1} f' = 1 + v f''$ where $f', f'' \in \mathbf{Z}[v]$. Moreover, $\sum_{j \in [1,r]} f_{j,c}^2 > 0$. It follows that $c = 0$ and $\sum_{j \in [1,r]} f_{j,0}^2 = 1$ so that there exists $j_0 \in [1, r]$ such that $f_{j_0,0} = \pm 1$ and $f_{j,0} = 0$ for $j \neq j_0$. We have $f_j = \pm \delta_{j,j_0} \pmod{v\mathbf{Z}[v]}$ for all j . Since $\bar{f}_j = f_j$ we deduce that $f_j = \pm \delta_{j,j_0}$ for all j . Thus $b = \pm B_{j_0}$. This completes the proof of (a). At the same time we have proved (b). Clearly, $\{B_1, B_2, \dots, B_r\}$ has the positivity property in (c) (with $\mathbf{V}_{\mathcal{A}}$ identified with ${}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ and with $a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}}$ identified with $\gamma(a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}})$). Since any B_j appears with > 0 coefficient in some $\gamma(a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}})$, we see that $\{B_1, B_2, \dots, B_r\}$ is the only basis contained in $\{\pm B_1, \pm B_2, \dots, \pm B_r\}$ with the positivity property in (c). This completes the proof of the proposition.

11.12. From the proof of 11.11 we see that $\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ (see Proposition 10.19(a)) restricts to a bijection

$$(a) \quad \mathbf{B} \xrightarrow{\sim} {}^{\xi}\mathfrak{B}.$$

For any G_0 -orbit \mathcal{O} in $\mathfrak{g}_{\delta}^{nil}$ let $\mathbf{B}_{\mathcal{O}}$ be the set of all $b \in \mathbf{B}$ such that $\gamma(b) \in \mathfrak{B}_{\mathcal{O}}$ (see 10.13). We have a partition $\mathbf{B} = \sqcup_{\mathcal{O}} \mathbf{B}_{\mathcal{O}}$ where \mathcal{O} runs over the G_0 -orbits in $\mathfrak{g}_{\delta}^{nil}$.

11.13. We consider the setup of 10.22. We show:

(a) *The \mathcal{A} -submodule $\mathbf{V}_{\mathcal{A}}$ of \mathbf{V} defined in 11.2 in terms of $\dot{\xi}$ is the same as that defined in terms of $\dot{\xi}_1$.*

It is not clear how to prove this using the definition in 11.2 since \mathbf{E}'' defined in terms of $\dot{\xi}$ (see 10.3) is not necessarily the same as that defined in terms of $\dot{\xi}_1$. Instead we will argue indirectly. Using 11.4(g) it is enough to show:

(b) *The isomorphism $\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ defined in 10.19 in terms of $\dot{\xi}$ is equal to the analogous isomorphism defined in terms of $\dot{\xi}_1$.*

Thus it is enough to show that if $\varpi \in \mathbf{E}' \cap \mathbf{E}'_1$, then \tilde{I}_{ϖ} defined in 10.2 in terms of $\dot{\xi}$ is the same as that defined in terms of $\dot{\xi}_1$. Using the definitions we see that it is enough to show that

$$\begin{aligned} \dot{\eta}_1 \mathfrak{p}_{\eta}^{(|\eta|/2)(\varpi+\iota)} &= \dot{\eta}_1 \mathfrak{p}_{\eta_1}^{(|\eta_1|/2)(\varpi+\iota)}, \\ \dot{\eta} \mathfrak{p}_0^{(|\eta|/2)(\varpi+\iota)} &= \dot{\eta}_1 \mathfrak{p}_0^{(|\eta_1|/2)(\varpi+\iota)}. \end{aligned}$$

or that

$$\begin{aligned} \oplus_{\kappa \in \mathbf{Q}; \kappa \geq |\eta|} \binom{(|\eta|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta} &= \oplus_{\kappa \in \mathbf{Q}; \kappa \geq |\eta_1|} \binom{(|\eta_1|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta}, \\ \oplus_{\kappa \in \mathbf{Q}; \kappa \geq 0} \binom{(|\eta|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta} &= \oplus_{\kappa \in \mathbf{Q}; \kappa \geq 0} \binom{(|\eta_1|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta}, \end{aligned}$$

or that

$$\begin{aligned} \oplus_{\kappa \in \mathbf{Q}; \kappa/|\eta| \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta|} \mathfrak{g}_{\delta} &= \oplus_{\kappa \in \mathbf{Q}; \kappa/|\eta_1| \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta_1|} \mathfrak{g}_{\delta}, \\ \oplus_{\kappa \in \mathbf{Q}; \kappa/|\eta| \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta|} \mathfrak{g}_{\delta} &= \oplus_{\kappa \in \mathbf{Q}; \kappa/|\eta_1| \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta_1|} \mathfrak{g}_{\delta}, \end{aligned}$$

or, setting $\kappa' = \kappa/|\eta|$, $\kappa'' = \kappa/|\eta_1|$, that

$$\begin{aligned} \oplus_{\kappa' \in \mathbf{Q}; \kappa' \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa'} \mathfrak{g}_{\delta} &= \oplus_{\kappa'' \in \mathbf{Q}; \kappa'' \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa''} \mathfrak{g}_{\delta}, \\ \oplus_{\kappa' \in \mathbf{Q}; \kappa' \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa'} \mathfrak{g}_{\delta} &= \oplus_{\kappa'' \in \mathbf{Q}; \kappa'' \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa''} \mathfrak{g}_{\delta}, \end{aligned}$$

which are obvious. This proves (a).

Using (b) and 11.12(b) we see that the basis \mathbf{B} of $\mathbf{V}_{\mathcal{A}}$ defined in 11.11 in terms of $\dot{\xi}$ is the same as that defined in terms of $\dot{\xi}_1$.

12. PURITY PROPERTIES

In this section we show that for any irreducible local system \mathcal{L} on a G_0 -orbit in $\mathfrak{g}_{\delta}^{nil}$ the cohomology sheaves of $\mathcal{L}^{\sharp} \in \mathcal{D}(\mathfrak{g}_{\delta})$ satisfy a strong purity property. This generalizes the analogous result in the \mathbf{Z} -graded case in [L4].

12.1. In this section we assume that $p > 0$ and that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q with q elements (here q is a power of p). Replacing q by larger powers of p if necessary, we can assume that m divides $q - 1$ and that we can find an \mathbf{F}_q -rational structure on G with Frobenius map $F : G \rightarrow G$ such that $\vartheta : G \rightarrow G$ (see 0.5) commutes with $F : G \rightarrow G$. Then G_0 is defined over \mathbf{F}_q and \mathfrak{g} inherits from G an \mathbf{F}_q -rational structure with Frobenius map $F : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $F(\mathfrak{g}_i) = \mathfrak{g}_i$ for all i . Again by replacing q by larger powers of p if necessary, we may assume that all G_0 -orbits in $\mathfrak{g}_{\delta}^{nil}$ are defined over \mathbf{F}_q and that for any irreducible G_0 -equivariant local system \mathcal{L} on a G_0 -orbit \mathcal{O} in $\mathfrak{g}_{\delta}^{nil}$ we have $F^* \mathcal{L} \cong \mathcal{L}$.

We now fix a G_0 -orbit \mathcal{O} in $\mathfrak{g}_{\delta}^{nil}$ with closure $\bar{\mathcal{O}}$ and an irreducible G_0 -equivariant local system \mathcal{L} on \mathcal{O} . We fix an isomorphism $\tilde{F} : F^* \mathcal{L} \rightarrow \mathcal{L}$ which induces the identity map on the stalk of \mathcal{L} at some point of \mathcal{O}^F . Then \tilde{F} induces an isomorphism (denoted again by \tilde{F}) $F^* \mathcal{L}^{\sharp} \rightarrow \mathcal{L}^{\sharp}$. Given a finite-dimensional $\bar{\mathbf{Q}}_l$ -vector space V

with an endomorphism $\tilde{F} : V \rightarrow V$, we say that $\tilde{F} : V \rightarrow V$ is a -pure (for an integer a) if the eigenvalues of \tilde{F} are algebraic numbers all of whose complex conjugates have absolute value $q^{a/2}$. Sometimes we will just say that V is a -pure (where this is understood to refer to \tilde{F}).

12.2. We show:

(a) *For any $x \in \bar{\mathcal{O}}^F$ and any $a \in \mathbf{Z}$, the induced linear map $\tilde{F} : \mathcal{H}_x^a(\mathcal{L}^\sharp) \rightarrow \mathcal{H}_x^a(\mathcal{L}^\sharp)$ is a -pure.*

Using [LY, 2.3(b)], we can find $\phi = (e, h, f) \in J_\delta(x)$ such that h, f are \mathbf{F}_q -rational. Recall that $x = e$. Let $\iota = \iota_\phi \in Y_G$ be as in 1.1. Let $\mathfrak{z}(f)$ be the centralizer of f in \mathfrak{g} and let $\Sigma = e + \mathfrak{z}(f) \subset \mathfrak{g}$. According to Slodowy:

(b) *The affine space Σ is a transversal slice at x to the G -orbit of e in \mathfrak{g} and the \mathbf{k}^* -action $t \mapsto t^{-2} \text{Ad}(\iota(t))$ on \mathfrak{g} keeps e fixed, leaves Σ stable and defines a contraction of Σ to x .*

Let $\tilde{\Sigma} = \Sigma \cap \mathfrak{g}_\delta = e + (\mathfrak{z}(f) \cap \mathfrak{g}_\delta)$. Then:

(c) *$\tilde{\Sigma}$ is a transversal slice to the G_0 -orbit of e in \mathfrak{g}_δ ; the \mathbf{k}^* -action in (b) leaves stable $\tilde{\Sigma}$ and is a contraction of $\tilde{\Sigma}$ to e .*

(We have a direct sum decomposition $\mathfrak{g}_\delta = (\mathfrak{z}(f) \cap \mathfrak{g}_\delta) \oplus [e, \mathfrak{g}_0]$ obtained by taking the ζ^δ -eigenspace of $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ in the two sides of the direct sum decomposition $\mathfrak{g} = \mathfrak{z}(f) \oplus [e, \mathfrak{g}]$; note that both $\mathfrak{z}(f)$ and $[e, \mathfrak{g}]$ are θ -stable.)

Let \mathcal{L}' be the restriction of \mathcal{L} to $\mathcal{O} \cap \tilde{\Sigma}$ (a smooth irreducible subvariety of $\tilde{\Sigma}$). Note that $\mathcal{O} \cap \tilde{\Sigma}$, $\bar{\mathcal{O}} \cap \tilde{\Sigma}$ are stable under the \mathbf{k}^* -action in (c) and \mathcal{L}' is equivariant for that action. By (c), we have $\mathcal{H}_x(\mathcal{L}^\sharp) = \mathcal{H}_x(\mathcal{L}'^\sharp)$. It remains to show that $\mathcal{H}_x(\mathcal{L}'^\sharp)$ is a -pure. This can be reduced to Deligne's hard Lefschetz theorem by an argument in Lemma 4.5(b) in [KL2] applied to $\bar{\mathcal{O}} \cap \tilde{\Sigma} \subset \tilde{\Sigma}$ with the contraction in (c) and to \mathcal{L}' instead of \mathcal{Q}_t . Note that in [KL2, 4.5(b)] an inductive purity assumption was made which is in fact unnecessary, by Gabber purity theorem. This completes the proof of (a).

Erratum to [L4]. On page 209, line -5, replace $t^{-n} \text{Ad}(\iota'(t))$ by $t^{-2} \text{Ad}(\iota'(t))$.

13. AN INNER PRODUCT

This section is an adaptation of [L4, §19] from the \mathbf{Z} -graded case to the \mathbf{Z}/m -graded case; we express the matrix whose entries are the values of the $(:)$ -pairing at two elements of \mathfrak{B} as a product of three matrices. As an application we show (see 13.8(a)) that if $(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')$ in $\mathcal{I}(\mathfrak{g}_\delta)$ are such that some cohomology sheaf of $\mathcal{L}^\sharp|_{\mathcal{O}'}$ contains \mathcal{L}' , then $(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')$ are in the same block. Another application (to odd vanishing) is given in Section 14.

13.1. We fix $(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')$ in $\mathcal{I}(\mathfrak{g}_\delta)$ and we form $A = \mathcal{L}^\sharp, A' = \mathcal{L}'^\sharp$ in $\mathcal{D}(\mathfrak{g}_\delta)$. We want to compute $d_j(\mathfrak{g}_\delta; A, A')$ (see [LY, 0.12]) for a fixed $j \in \mathbf{Z}$. We can arrange the G_0 -orbits in $\mathfrak{g}_\delta^{nil}$ in an order $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_\beta$ such that $\mathcal{O}_{\leq s} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \dots \cup \mathcal{O}_s$ is closed in \mathfrak{g}_δ for $s = 0, 1, \dots, \beta$. We choose a smooth irreducible variety Γ with a free action of G_0 such that $H^r(\Gamma, \bar{\mathcal{Q}}_t) = 0$ for $r = 1, 2, \dots, \mathbf{m}$ where \mathbf{m} is a large integer (compared to j). We assume that $\mathbf{D} := \dim \Gamma$ is large (compared to j). We have $H_c^{2\mathbf{D}-i}(\Gamma, \bar{\mathcal{Q}}_t) = 0$ for $i = 1, 2, \dots, \mathbf{m}$. We form $X = G_0 \backslash (\Gamma \times \mathfrak{g}_\delta)$. Let $\tilde{\mathcal{L}}, \tilde{\mathcal{L}}'$ be the local systems on the smooth subvarieties $G_0 \backslash (\Gamma \times \mathcal{O}), G_0 \backslash (\Gamma \times \mathcal{O}')$ of X defined by $\mathcal{L}, \mathcal{L}'$ and let $\tilde{A} = \tilde{\mathcal{L}}^\sharp, \tilde{A}' = \tilde{\mathcal{L}}'^\sharp$ be the corresponding intersection cohomology complexes in $\mathcal{D}(X)$. Then $\tilde{A} \otimes \tilde{A}' \in \mathcal{D}(X)$ is well defined; its restriction to various

subvarieties of X will be denoted by the same symbol. For $s = 0, 1, \dots, \beta$ we form $X_s = G_{\underline{0}} \backslash (\Gamma \times \mathcal{O}_s)$, $X_{\leq s} = G_{\underline{0}} \backslash (\Gamma \times \mathcal{O}_{\leq s})$. We set $X_{\leq -1} = \emptyset$. The partition of $X_{\leq s}$ into $X_{\leq s-1}$ and X_s (for $s = 0, 1, \dots, \beta$) gives rise to a long exact sequence

$$(a) \quad \begin{aligned} \dots &\xrightarrow{\xi_a} H_c^a(X_s, \tilde{A} \otimes \tilde{A}') \rightarrow H_c^a(X_{\leq s}, \tilde{A} \otimes \tilde{A}') \\ &\rightarrow H_c^a(X_{\leq s-1}, \tilde{A} \otimes \tilde{A}') \xrightarrow{\xi_{a+1}} H_c^{a+1}(X_s, \tilde{A} \otimes \tilde{A}') \rightarrow \dots \end{aligned}$$

13.2. In the following proposition we encounter two kinds of integers; some like j , m , $\dim G_{\underline{0}}$, $\dim \mathfrak{g}_{\delta}$, β are regarded as “small” (they belong to a fixed finite set of integers), others like $2\mathbf{D}$ and \mathbf{m} are regarded as very large (we are free to choose them so). We will also encounter integers a such that $2\mathbf{D} - a$ is a “small” integer (we then write $a \sim 2\mathbf{D}$).

Proposition 13.3.

(a) Assume that \mathbf{k} is as in 12.1. Then $H_c^a(X_s, \tilde{A} \otimes \tilde{A}')$ is a -pure (see 12.1) for $s = 0, 1, \dots, \beta$ and $a \sim 2\mathbf{D}$.

(b) We choose $x_s \in \mathcal{O}_s$. If $a \sim 2\mathbf{D}$, then

$$\begin{aligned} &\dim H_c^a(X_s, \tilde{A} \otimes \tilde{A}') \\ &= \sum_{r+r_1+r_2=a} \dim(H_c^r(G_{\underline{0}}(x_s)^0 \backslash \Gamma, \bar{\mathbf{Q}}_l) \otimes \mathcal{H}_{x_s}^{r_1} A \otimes \mathcal{H}_{x_s}^{r_2} A')^{G_{\underline{0}}(x_s)/G_{\underline{0}}(x_s)^0}, \end{aligned}$$

where the upper script refers to invariants under the finite group $G_{\underline{0}}(x_s)/G_{\underline{0}}(x_s)^0$.

(c) Assume that \mathbf{k} is as in 12.1. Then $H_c^a(X_{\leq s}, \tilde{A} \otimes \tilde{A}')$ is a -pure (see 12.1) for $s = 0, 1, \dots, \beta$ and $a \sim 2\mathbf{D}$.

(d) The exact sequence 13.1(a) gives rise to short exact sequences

$$0 \rightarrow H_c^a(X_s, \tilde{A} \otimes \tilde{A}') \rightarrow H_c^a(X_{\leq s}, \tilde{A} \otimes \tilde{A}') \rightarrow H_c^a(X_{\leq s-1}, \tilde{A} \otimes \tilde{A}') \rightarrow 0$$

for $s = 0, 1, \dots, \beta$ and $a \sim 2\mathbf{D}$.

(e) For $a \sim 2\mathbf{D}$ we have $\dim H_c^a(X, \tilde{A} \otimes \tilde{A}') = \sum_{s=0}^{\beta} \dim H_c^a(X_s, \tilde{A} \otimes \tilde{A}')$.

(f) For $a \sim 2\mathbf{D}$ we have

$$\begin{aligned} &\dim H_c^a(X, \tilde{A} \otimes \tilde{A}') \\ &= \sum_{s=0}^{\beta} \sum_{r+r_1+r_2=a} \dim(H_c^r(G_{\underline{0}}(x_s)^0 \backslash \Gamma, \bar{\mathbf{Q}}_l) \otimes \mathcal{H}_{x_s}^{r_1} A \otimes \mathcal{H}_{x_s}^{r_2} A')^{G_{\underline{0}}(x_s)/G_{\underline{0}}(x_s)^0}. \end{aligned}$$

The proof is almost a copy of the proof of [L4, 19.4]. By general principles we can assume that \mathbf{k} is as in 12.1. We shall use Deligne’s theory of weights. We first prove (a) and (b). We write x instead of x_s . We may assume that x is an \mathbf{F}_q -rational point. We have a natural spectral sequence

$$(g) \quad E_2^{r,r'} = H_c^r(X_s, \mathcal{H}^{r'}(\tilde{A} \otimes \tilde{A}')) \implies H_c^{r+r'}(X_s, \tilde{A} \otimes \tilde{A}').$$

We show that

$$(h) \quad E_2^{r,r'} \text{ is } (r+r')\text{-pure if } r+r' \sim 2\mathbf{D}.$$

We have $X_s = G_{\underline{0}}(x) \backslash \Gamma$ and

$$E_2^{r,r'} = (H_c^r(G_{\underline{0}}(x)^0 \backslash \Gamma, \bar{\mathbf{Q}}_l) \otimes \mathcal{H}_x^{r'}(A \otimes A'))^{G_{\underline{0}}(x)/G_{\underline{0}}(x)^0}.$$

Here $A \otimes A' \in \mathcal{D}(\mathfrak{g}_\delta)$. We may assume that r' is “small” (otherwise, $E_2^{r,r'} = 0$). We then have $r \sim 2\mathbf{D}$. Now

$$\mathcal{H}_x^{r'}(A \otimes A') = \oplus_{r_1+r_2=r'} \mathcal{H}_x^{r_1}(A) \otimes \mathcal{H}_x^{r_2}(A')$$

is r' -pure by 12.2(a). Moreover, $H_c^r(G_0(x)^0 \setminus \Gamma, \bar{\mathbf{Q}}_l)$ is r -pure for $r \sim 2\mathbf{D}$ (a known property of the classifying space of $G_0(x)^0$) and (h) follows.

From (h) it follows that $E_\infty^{r,r'}$ of the spectral sequence (g) is $(r + r')$ -pure if $r + r' \sim 2\mathbf{D}$ and (a) follows. From (h) it also follows that $E_2^{r,r'} = E_\infty^{r,r'}$ if $r + r' \sim 2\mathbf{D}$ (many differentials must be zero since they respect weights) and (b) follows.

Now (c) follows from (a) using the exact sequence 13.2(a) and induction on s . From (c) and (a) we see that the homomorphism ξ_{a+1} in 13.2(a) is between pure spaces of different weight. Since ξ_{a+1} preserve weights, it must be 0. Similarly, $\xi_a = 0$ in 13.2(a) hence (d) holds. Now (e) follows from (d) since the support of $\tilde{A} \otimes \tilde{A}'$ is contained in $X_{\leq \beta}$; (f) follows from (b),(e). This completes the proof of the proposition.

13.4. Using the definitions we see that 13.3(f) implies:

$$(a) \quad d_j(\mathfrak{g}_\delta; A, A') = \sum_s \sum_{-r_0+r_1+r_2=j-2p_s} \dim(H_{r_0}^{G_0(x_s)^0}(\text{pt}) \otimes \mathcal{H}_{x_s}^{r_1} A \otimes \mathcal{H}_{x_s}^{r_2} A')^{G_0(x_s)/G_0(x_s)^0},$$

where

$$(b) \quad p_s = \dim G_0 - \dim G_0(x_s) = \dim \mathcal{O}_s$$

and $H_{r_0}^{G_0(x_s)^0}(\text{pt})$ denotes equivariant homology of a point. (See [L6] for the definition of equivariant homology.)

13.5. Given $(\mathcal{O}, \mathcal{L}), (\tilde{\mathcal{O}}, \tilde{\mathcal{L}})$ in $\mathcal{I}(\mathfrak{g}_\delta)$ we define $\mu(\tilde{\mathcal{L}}, \mathcal{L}) \in \mathbf{Z}[v^{-1}]$ by

$$(a) \quad \mu(\tilde{\mathcal{L}}, \mathcal{L}) = \sum_a \mu(a; \tilde{\mathcal{L}}, \mathcal{L}) v^{-a},$$

where $\mu(a; \tilde{\mathcal{L}}, \mathcal{L})$ is the number of times $\tilde{\mathcal{L}}$ appears in a decomposition of the local system $\mathcal{H}^a(\mathcal{L}^\sharp)|_{\tilde{\mathcal{O}}}$ as a direct sum of irreducible local systems. Note that $\mu(\tilde{\mathcal{L}}, \mathcal{L})$ is zero unless $\tilde{\mathcal{O}}$ is contained in the closure of \mathcal{O} .

If \mathcal{E} is an irreducible G_0 -equivariant local system on \mathcal{O}_s , we denote by $\rho_{\mathcal{E}}$ the irreducible $G_0(x_s)/G_0(x_s)^0$ -module corresponding to \mathcal{E} . With this notation we can rewrite 13.4(a) as follows:

$$\begin{aligned} d_j(\mathfrak{g}_\delta; A, A') &= \sum_s \sum_{-r_0+r_1+r_2=j-2p_s} \\ &\sum_{\mathcal{E}, \mathcal{E}'} \mu(r_1; \mathcal{E}, \mathcal{L}) \mu(r_2; \mathcal{E}', \mathcal{L}') \dim(H_{r_0}^{G_0(x_s)^0}(\text{pt}) \otimes \rho_{\mathcal{E}} \otimes \rho_{\mathcal{E}'})^{G_0(x_s)/G_0(x_s)^0}, \end{aligned}$$

where $\mathcal{E}, \mathcal{E}'$ run over the isomorphism classes of irreducible G_0 -equivariant local systems on \mathcal{O}_s . This may be written in terms of power series in $\mathbf{Q}((v))$ as follows:

$$(b) \quad \{A, A'\} = \sum_{s=0}^{\beta} \sum_{\mathcal{E}, \mathcal{E}'} \mu(\mathcal{E}, \mathcal{L}) \Xi(\mathcal{E}, \mathcal{E}') \mu(\mathcal{E}', \mathcal{L}'),$$

where

$$(c) \quad \Xi(\mathcal{E}, \mathcal{E}') = \sum_{r_0 \geq 0} \dim(H_{r_0}^{G_0(x_s)^0}(\text{pt}) \otimes \rho_{\mathcal{E}} \otimes \rho_{\mathcal{E}'}^{G_0(x_s)/G_0(x_s)^0}) v^{r_0-2p_s} \in \mathbf{Q}((v)).$$

13.6. Let $B_1, B_2 \in \mathfrak{B}$. We write $B_1 = \mathcal{L}_1^\sharp[\dim \mathcal{O}_1] \in \mathcal{D}(\mathfrak{g}_\delta)$, $B_2 = \mathcal{L}_2^\sharp[\dim \mathcal{O}_2] \in \mathcal{D}(\mathfrak{g}_\delta)$ with $(\mathcal{O}_1, \mathcal{L}_1), (\mathcal{O}_2, \mathcal{L}_2)$ in $\mathcal{I}(\mathfrak{g}_\delta)$. We set

$$\begin{aligned} P_{B_2, B_1} &= \mu(\mathcal{L}_2, \mathcal{L}_1) \in \mathbf{N}[v^{-1}], \text{ (see 13.5);} \\ \tilde{P}_{B_2, B_1} &= P_{D(B_2), D(B_1)} \in \mathbf{N}[v^{-1}]; \\ \Lambda_{B_2, B_1} &= \Xi(\mathcal{L}_2, \mathcal{L}_1) \in \mathbf{Q}((v)), \text{ (see 13.5) if } \mathcal{O}_1 = \mathcal{O}_2; \\ \Lambda_{B_2, B_1} &= 0 \text{ if } \mathcal{O}_1 \neq \mathcal{O}_2; \\ \tilde{\Lambda}_{B_2, B_1} &= \Lambda_{B_2, D(B_1)} \in \mathbf{Q}((v)). \end{aligned}$$

Note that:

- (a) $P_{B_2, B_1} \neq 0 \implies \dim \mathcal{O}_2 \leq \dim \mathcal{O}_1$; $\tilde{P}_{B_2, B_1} \neq 0 \implies \dim \mathcal{O}_2 \leq \dim \mathcal{O}_1$;
- (b) if $\dim \mathcal{O}_2 = \dim \mathcal{O}_1$, then $P_{B_2, B_1} = \delta_{B_2, B_1}$, $\tilde{P}_{B_2, B_1} = \delta_{B_2, B_1}$;
- (c) $\tilde{\Lambda}_{B_2, B_1} = 0$ if $\mathcal{O}_1 \neq \mathcal{O}_2$.

Then 13.5(b) can be rewritten as follows:

$$\{A, A'\} = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \Lambda_{B_1, B_2} P_{B_2, B''},$$

where $B = A[\dim \mathcal{O}]$, $B'' = A'[\dim \mathcal{O}']$, or as

$$\{A, A'\} = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \Lambda_{B_1, D(B_2)} P_{D(B_2), D(B')},$$

where $B = A[\dim \mathcal{O}]$, $D(B') = A'[\dim \mathcal{O}']$, or as

$$\{A, A'\} = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \tilde{\Lambda}_{B_1, B_2} \tilde{P}_{B_2, B'},$$

where $B = A[\dim \mathcal{O}]$, $D(B') = A'[\dim \mathcal{O}']$. We have

$$\{A, A'\} = v^{-\kappa(B) - \kappa(B')} \{B, D(B')\} = v^{-\kappa(B) - \kappa(B')} (B : B'),$$

hence

$$(d) \quad v^{-\kappa(B) - \kappa(B')} (B : B') = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \tilde{\Lambda}_{B_1, B_2} \tilde{P}_{B_2, B'}$$

for any B, B' in \mathfrak{B} . (Here κ is as in 10.13.) We show:

(e) *The following three matrices with entries in $\mathbf{Q}((v))$ (indexed by $\mathfrak{B} \times \mathfrak{B}$) are invertible:*

- (i) *the matrix $((B : B'))$;*
- (ii) *the matrix $\mathcal{M} := (v^{-\kappa(B) - \kappa(B')} (B : B'))$;*
- (iii) *the matrix $\mathcal{T} := (\tilde{\Lambda}_{B, B'})$.*

The matrix in (i) is invertible since $(B : B') \in \delta_{B, B'} + v\mathbf{N}[[v]]$ for all B, B' ; see [LY, 0.12]. This implies immediately that \mathcal{M} is invertible. Now, by (d), we have $\mathcal{M} = \mathcal{S}\mathcal{T}\mathcal{S}'$ where \mathcal{S} (resp. \mathcal{S}') is the matrix indexed by $\mathfrak{B} \times \mathfrak{B}$ whose (B, B') -entry is $P_{B', B}$ (resp. $\tilde{P}_{B, B'}$). Since \mathcal{M} is invertible, it follows that \mathcal{T} is invertible. This proves (e).

13.7. We show:

(a) *If B, B' in \mathfrak{B} satisfy $\psi(B) \neq \psi(B')$ (ψ as in [LY, 6.6]), then $P_{B,B'} = 0$, $\tilde{P}_{B,B'} = 0$, $\tilde{\Lambda}_{B,B'} = 0$.*

We can find a function $\mathfrak{B} \rightarrow \mathbf{Z}$, $B \mapsto c_B$ such that for B, B' in \mathfrak{B} we have $c_B = c_{B'}$ if and only if $\psi(B) = \psi(B')$. From 13.6 we deduce

$$\begin{aligned} & v^{c_B - c_{B'}} v^{-\kappa(B) - \kappa(B')} (B : B') \\ &= \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} v^{c_B - c_{B_1}} P_{B_1, B} v^{c_{B_1} - c_{B_2}} \tilde{\Lambda}_{B_1, B_2} v^{c_{B_2} - c_{B'}} \tilde{P}_{B_2, B'}. \end{aligned}$$

When B, B' vary, this again can be expressed as the decomposition of the matrix $\tilde{\mathcal{M}} := (v^{c_B - c_{B'}} v^{-\kappa(B) - \kappa(B')} (B : B'))$ (indexed by $\mathfrak{B} \times \mathfrak{B}$) as a product of three matrices $\tilde{\mathcal{S}} \tilde{\mathcal{T}} \tilde{\mathcal{S}}'$, where $\tilde{\mathcal{S}}$ (resp. $\tilde{\mathcal{S}}'$) is the matrix indexed by $\mathfrak{B} \times \mathfrak{B}$ whose (B, B') -entry is $v^{c_B - c_{B'}} P_{B', B}$ (resp. $v^{c_B - c_{B'}} \tilde{P}_{B, B'}$) and $\tilde{\mathcal{T}}$ is the matrix indexed by $\mathfrak{B} \times \mathfrak{B}$ whose (B, B') -entry is $v^{c_B - c_{B'}} \tilde{\Lambda}_{B, B'}$. From [LY, 7.9(a)] we know that $(B : B') = 0$ unless $\psi(B) = \psi(B')$. Hence $v^{c_B - c_{B'}} (B : B') = (B : B')$ for all B, B' so that $\tilde{\mathcal{M}} = \mathcal{M}$. Thus we have

$$\tilde{\mathcal{S}} \tilde{\mathcal{T}} \tilde{\mathcal{S}}' = \mathcal{S} \mathcal{T} \mathcal{S}'.$$

Now by 13.6(c),(e), the matrix \mathcal{T} (and hence the matrix $\tilde{\mathcal{T}}$) belongs to a subgroup of GL_N ($N = \sharp(\mathfrak{B})$) of the form $GL_{N_1} \times \cdots \times GL_{N_k}$ where N_1, \dots, N_k are the sizes of the various subsets \mathfrak{B}_O ; moreover, by 13.6(a),(b), the matrix \mathcal{S} (hence the matrix $\tilde{\mathcal{S}}$) belongs to the unipotent radical of a parabolic subgroup of GL_N with Levi subgroup equal to the subgroup of GL_N considered above, while the matrix \mathcal{S}' (hence the matrix $\tilde{\mathcal{S}}'$) belongs to the unipotent radical of the opposite parabolic subgroup with the same Levi subgroup. This forces the equalities $\tilde{\mathcal{S}} = \mathcal{S}$, $\tilde{\mathcal{T}} = \mathcal{T}$, $\tilde{\mathcal{S}}' = \mathcal{S}'$. Now the equality $\tilde{\mathcal{S}} = \mathcal{S}$ implies $v^{c_B - c_{B'}} P_{B', B} = P_{B', B}$ for all B', B in \mathfrak{B} . Thus, if $\psi(B) \neq \psi(B')$ (so that $c_B \neq c_{B'}$), we must have $P_{B', B} = 0$. Similarly, from $\tilde{\mathcal{T}} = \mathcal{T}$ we see that, if $\psi(B) \neq \psi(B')$, then $\tilde{\Lambda}_{B, B'} = 0$ and from $\tilde{\mathcal{S}}' = \mathcal{S}'$ we see that, if $\psi(B) \neq \psi(B')$, then $\tilde{P}_{B, B'} = 0$. This proves (a).

13.8. We now fix $\xi \in \underline{\mathfrak{T}}_\eta$. We define four matrices ${}^\xi \mathcal{M}, {}^\xi \mathcal{S}, {}^\xi \mathcal{T}, {}^\xi \mathcal{S}'$ indexed by ${}^\xi \mathfrak{B} \times {}^\xi \mathfrak{B}$ as follows. For B, B' in ${}^\xi \mathfrak{B}$, the (B, B') -entry of ${}^\xi \mathcal{M}$ is $v^{-\kappa(B) - \kappa(B')} (B : B')$; the (B, B') -entry of ${}^\xi \mathcal{S}$ is $P_{B', B}$; the (B, B') -entry of ${}^\xi \mathcal{T}$ is $\tilde{\Lambda}_{B, B'}$; the (B, B') -entry of ${}^\xi \mathcal{S}'$ is $\tilde{P}_{B, B'}$. Using 13.7(a) we deduce from 13.6(d) the equality of matrices

$$(a) \quad {}^\xi \mathcal{M} = ({}^\xi \mathcal{S}) ({}^\xi \mathcal{T}) ({}^\xi \mathcal{S}').$$

As in the proof of 13.7(a) the last equality determines uniquely the matrices ${}^\xi \mathcal{S}, {}^\xi \mathcal{T}, {}^\xi \mathcal{S}'$ if the matrix ${}^\xi \mathcal{M}$ is known; in fact, it provides an algorithm for computing the entries of these three matrices (and in particular for the entries $P_{B', B}$ in terms of the entries of ${}^\xi \mathcal{M}$. Now under the bijection $\gamma : \mathbf{B} \xrightarrow{\sim} {}^\xi \mathfrak{B}$ (see 11.12(a)) the matrix ${}^\xi \mathcal{M}$ becomes a matrix indexed by $\mathbf{B} \times \mathbf{B}$ whose (b, b') -entry is $v^{-\kappa(B) - \kappa(B')} (b, b')$; these entries are explicitly computable from the combinatorial description of $(:)$ on \mathbf{V} . We see that:

(b) *The quantities $P_{B', B}$ are computable by an algorithm provided that the bijection $\gamma : \mathbf{B} \xrightarrow{\sim} {}^\xi \mathfrak{B}$ is known.*

This can be used in several examples to compute the $P_{B', B}$ explicitly. The algorithm in (b) seems to depend on the choice of η such that $\underline{\eta} = \delta$; but in fact, by the results in [LY, 3.9], 10.22, 11.13, it does not depend on this choice.

In the case where $m = 1$, there is another algorithm to compute the quantities $P_{B',B}$; see [L7, Theorem 24.8]. It again displays the matrix ${}^\xi\mathcal{S}$ as the first of the three factors in a matrix decomposition like (a), but with the matrix ${}^\xi\mathcal{M}$ being replaced by a matrix indexed by a pair of irreducible representation of a Weyl group and with entries determined by a prescription quite different from that used in this paper. In that case the bijection $\gamma : \mathbf{B} \xrightarrow{\sim} {}^\xi\mathfrak{B}$ is replaced by the “generalized Springer correspondence”. It would be interesting to understand better the connection between these two approaches to the quantities $P_{B',B}$.

14. ODD VANISHING

In this section we show that for any irreducible local system \mathcal{L} on a G_0 -orbit in $\mathfrak{g}_\delta^{\text{nil}}$ the cohomology sheaves of $\mathcal{L}^\sharp \in \mathcal{D}(\mathfrak{g}_\delta)$ are zero in odd degrees. (See Theorem 14.10.) In the case where $m \gg 0$, this follows from the analogous result in the \mathbf{Z} -graded case in [L4]. In the case where $m = 1$ this follows from [L7, Theorem 24.8(a)]. In the case where $m > 1, \delta = \underline{1}$, $\mathcal{L} = \bar{\mathbf{Q}}_l$ and $G, \mathfrak{g} = \oplus_i \mathfrak{g}_i$ are as in [LY, 0.3], this follows from [L8, Theorem 11.3] and from the known odd vanishing result for affine Schubert varieties.

14.1. We preserve the setup of 10.1, 10.2. For $\varpi \in \mathbf{E}'$ let $h(\varpi) \in \mathbf{Z}$ be as in 10.2. For $\mathbf{c} \in \mathring{\underline{\mathbf{E}}}$ we set $h(\mathbf{c}) = h(\varpi)$ where ϖ is any element of $\mathbf{c} \cap \mathbf{E}'$; this is well defined, by 10.7(a). For $\mathbf{c} \in \mathring{\underline{\mathbf{E}}}$ we set $T_{\mathbf{c}} = v^{-h(\mathbf{c})} \tilde{T}_{\mathbf{c}} \in \mathbf{V}'$. Note that $\{T_{\mathbf{c}}; \mathbf{c} \in \mathring{\underline{\mathbf{E}}}\}$ is a $\mathbf{Q}(v)$ -basis of \mathbf{V}' . Let $f \mapsto f^\heartsuit$ be the field involution of $\mathbf{Q}(v)$ which carries v to $-v$; this extends to a field involution of $\mathbf{Q}((v))$ (denoted again by $f \mapsto f^\heartsuit$) given by $\sum_i c_i v^i \mapsto \sum_i c_i (-v)^i$, where $c_i \in \mathbf{Q}$.

Let $b \mapsto b^\heartsuit$ be the \mathbf{Q} -linear involution $\mathbf{V}' \rightarrow \mathbf{V}'$ such that $(fT_{\mathbf{c}})^\heartsuit = f^\heartsuit T_{\mathbf{c}}$ for any $\mathbf{c} \in \mathring{\underline{\mathbf{E}}}$ and $f \in \mathbf{Q}(v)$.

Lemma 14.2 (14.2). *For any b, b' in \mathbf{V}' we have $(b^\heartsuit : b'^\heartsuit) = (b : b')^\heartsuit$.*

It is enough to show that for any \mathbf{c}, \mathbf{c}' in $\mathring{\underline{\mathbf{E}}}$ and any f, f' in $\mathbf{Q}(v)$ we have

$$(f^\heartsuit T_{\mathbf{c}} : f'^\heartsuit T_{\mathbf{c}'}) = (fT_{\mathbf{c}} : f'T_{\mathbf{c}'})^\heartsuit.$$

We can assume that $f = f' = 1$. It is enough to show that

$$(a) \quad (T_{\mathbf{c}} : T_{\mathbf{c}'}) \in \mathbf{Q}(v^2),$$

or that

$$v^{-h(\varpi_1)+h(\varpi_2)} [\varpi_1 | \varpi_2] \in \mathbf{Q}(v^2)$$

for any ϖ_1, ϖ_2 in \mathbf{E}' (notation of 10.12(a)) or that

$$(b) \quad v^{-h(\varpi_1)+h(\varpi_2)} \sum_{w \in \mathcal{W}} v^{\tau(\varpi_2, w\varpi_1)} \in \mathbf{Q}(v^2).$$

From 10.8(b) we see that for ϖ, ϖ' in \mathbf{E}' we have

$$(b) \quad \tau(\varpi, \varpi') = h(\varpi) + h(\varpi') \pmod{2}.$$

From 10.11(b) we see that for $\varpi \in \mathbf{E}'$, $w \in \mathcal{W}$ and $N \in \mathbf{Z}$ we have

$$\dim {}^\epsilon \mathfrak{p}_N^{w\varpi} = \dim {}^\epsilon \mathfrak{p}_N^{\varpi}, \text{ hence } \dim {}^\epsilon \mathfrak{u}_N^{w\varpi} = \dim {}^\epsilon \mathfrak{u}_N^{\varpi},$$

so that

$$h(w\varpi) = h(\varpi).$$

Using this and (b) we see that for ϖ, ϖ' in \mathbf{E}' we have

$$\sum_{w \in \mathcal{W}} v^{\tau(\varpi', w\varpi)} \in v^{h(\varpi') + h(\varpi)} \mathbf{N}[v^2, v^{-2}].$$

Thus, (b) holds. The lemma is proved.

14.3. From 14.2 we deduce that $\heartsuit : \mathbf{V}' \rightarrow \mathbf{V}'$ carries $\mathfrak{R}_l = \mathfrak{R}_r$ onto $\mathfrak{R}_l = \mathfrak{R}_r$; hence it induces a \mathbf{Q} -linear involution $\mathbf{V} \rightarrow \mathbf{V}$ (denoted again by \heartsuit). From 14.2 we deduce:

(a) For any b, b' in \mathbf{V} we have $(b^\heartsuit : b'^\heartsuit) = (b : b')^\heartsuit$.

14.4. Let $\mathbf{c} \in \mathring{\mathbf{E}}$, $\varpi \in \mathbf{E}'' \cap \mathbf{c}$ and let $a_\varpi = \sum_j v^{-2s_j} v^{d(\varpi)} \in \mathcal{A}$ be as in 11.1. Note that

$$(a_\varpi)^\heartsuit = \sum_j v^{-2s_j} (-v)^{d(\varpi)} = (-1)^{d(\varpi)} a_\varpi.$$

Hence we have

$$(a_\varpi^{-1} \tilde{T}_{\mathbf{c}})^\heartsuit = (a_\varpi^{-1} v^{h(\mathbf{c})} T_{\mathbf{c}})^\heartsuit = (-1)^{d(\varpi) + h(\mathbf{c})} a_\varpi^{-1} v^{h(\mathbf{c})} T_{\mathbf{c}} = (-1)^{d(\varpi) + h(\mathbf{c})} a_\varpi^{-1} \tilde{T}_{\mathbf{c}}.$$

It follows that:

(a) $\heartsuit : \mathbf{V}' \rightarrow \mathbf{V}'$ carries $\mathbf{V}'_{\mathcal{A}}$ onto $\mathbf{V}'_{\mathcal{A}}$ and $\heartsuit : \mathbf{V} \rightarrow \mathbf{V}$ carries $\mathbf{V}_{\mathcal{A}}$ onto $\mathbf{V}_{\mathcal{A}}$.

14.5. We show:

a) For any $b \in \mathbf{V}'$ we have $\overline{b^\heartsuit} = (\bar{b})^\heartsuit$. Hence for any $b \in \mathbf{V}$ we have $\overline{b^\heartsuit} = (\bar{b})^\heartsuit$.

We can assume that $f = f T_{\mathbf{c}}$ where $\mathbf{c} \in \mathring{\mathbf{E}}$ and $f \in \mathbf{Q}(v)$. Note that $\overline{f^\heartsuit} = (\bar{f})^\heartsuit$. Hence we can assume that $f = 1$. We have

$$\overline{T_{\mathbf{c}}} = \overline{v^{-h(\mathbf{c})} \tilde{T}_{\mathbf{c}}} = v^{h(\mathbf{c})} \tilde{T}_{\mathbf{c}} = v^{2h(\mathbf{c})} T_{\mathbf{c}},$$

hence

$$(\overline{T_{\mathbf{c}}})^\heartsuit = v^{2h(\mathbf{c})} T_{\mathbf{c}}.$$

We have $\overline{T_{\mathbf{c}}^\heartsuit} = \overline{T_{\mathbf{c}}} = v^{2h(\mathbf{c})} T_{\mathbf{c}}$. This proves (a).

14.6. We show:

(a) $b \mapsto b^\heartsuit$ is a bijection $\mathbf{B}' \xrightarrow{\sim} \mathbf{B}'$.

Let $b \in \mathbf{B}'$. From $b \in \mathbf{V}_{\mathcal{A}}$ we see using 14.4(a) that $b^\heartsuit \in \mathbf{V}_{\mathcal{A}}$. From $\bar{b} = b$ we see using 14.5(a) that $\overline{b^\heartsuit} = (\bar{b})^\heartsuit = b^\heartsuit$. From $(b : b) \in \mathbf{Q}(v) \cap (1 + v\mathbf{Z}[[v]])$ we see using 14.3(a) that

$$(b^\heartsuit : b'^\heartsuit) = (b : b')^\heartsuit \in (\mathbf{Q}(v) \cap (1 + v\mathbf{Z}[[v]]))^\heartsuit = \mathbf{Q}(v) \cap (1 + v\mathbf{Z}[[v]]) \subset 1 + v\mathbf{Z}[[v]].$$

Using this and the definitions, we see that $b^\heartsuit \in \mathbf{B}'$. Thus the map $b \mapsto b^\heartsuit$, $\mathbf{B}' \rightarrow \mathbf{B}'$ is well defined. Since this map has square 1, it is a bijection. This proves (a).

From (a) we deduce:

(b) If $b \in \mathbf{B}$, then $b^\heartsuit = s_b \tilde{b}$ for a well-defined $s_b \in \{1, -1\}$ and a well-defined $\tilde{b} \in \mathbf{B}$.

The following result makes (b) more precise.

Lemma 14.7 (14.7). *Let \mathcal{O} be a G_0 -orbit in $\mathfrak{g}_\delta^{nil}$ and let $b \in \mathbf{B}_{\mathcal{O}}$. We have $b^\heartsuit = (-1)^{\dim \mathcal{O}_b} b$.*

We argue by induction on $\dim \mathcal{O}$; we can assume that the result holds when \mathcal{O} is replaced by an orbit of dimension $< \dim \mathcal{O}$ (if any). We have $\gamma(b) = \mathcal{L}^\#[\dim \mathcal{O}]$ where $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$. Let $x \in \mathcal{O}$; we associate to x an ϵ -spiral \mathfrak{p}_*^ϕ and a splitting $\tilde{\mathfrak{l}}_*^\phi$ of it as in [LY, 2.9]. Recall that $\tilde{\mathfrak{l}}_\eta^\phi \subset \mathcal{O}$ and that $\mathcal{L}_1 := \mathcal{L}|_{\tilde{\mathfrak{l}}_\eta^\phi}$ is an irreducible \tilde{L}_0^ϕ -equivariant local system on $\tilde{\mathfrak{l}}_\eta^\phi$; thus, $\mathcal{L}_1^\# \in \mathcal{D}(\tilde{\mathfrak{l}}_\eta^\phi)$ is defined. Let $I = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta^\phi}^{\mathfrak{g}_\delta}(\mathcal{L}_1^\#) \in \mathcal{Q}(\mathfrak{g}_\delta)$; we have clearly $I \in {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$. By [LY, 2.9(b),(c)], in ${}^\xi \mathcal{K}(\mathfrak{g}_\delta)$ we have

$$I = \mathcal{L}^\# + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'} \gamma(b'),$$

where $f_{b'} \in \mathcal{A}$. Define $I' \in \mathbf{V}_{\mathcal{A}}$ by $\gamma(I') = I$. We have

$$(a) \quad I' = v^{-\dim \mathcal{O}} b + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'} b'.$$

By [L4, 17.2, 17.3], in $\mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)$ we have

$$(b) \quad h\mathcal{L}_1^\# = \sum_{j \in J} h_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(C(j)),$$

where $h = h^\heartsuit \in \mathcal{A} - \{0\}$, J is a finite set and for $j \in J$, $\mathfrak{q}(j)$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}_\eta^\phi$ with Levi subalgebra $\mathfrak{m}(j)$ such that $\mathfrak{q}(j), \mathfrak{m}(j)$ are compatible with the \mathbf{Z} -grading of $\tilde{\mathfrak{l}}_\eta^\phi$, $C(j) \in \mathcal{Q}(\mathfrak{m}(j)_\eta)$ is a cuspidal perverse sheaf and $h_j \in \mathcal{A}$. Moreover, since $\mathcal{L}_1^\#$ belongs to the block of $\mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$ given by ξ , we can assume that $\mathfrak{m}(j) = \mathfrak{m}$ and $C(j) = \tilde{C}$ for all j . Thus (b) can be written in the form

$$(c) \quad h\mathcal{L}_1^\# = F_0 + F_1,$$

where

$$F_0 = \sum_{j \in J} h'_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta])),$$

$$F_1 = \sum_{j \in J} h''_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta])),$$

$$h'_j + h''_j = h_j v^{\dim \mathfrak{m}_\eta}, h'_j \in \mathbf{Z}[v^2, v^{-2}], h''_j \in v\mathbf{Z}[v^2, v^{-2}].$$

Let $\mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev}$ be the $\mathbf{Z}[v^2, v^{-2}]$ -submodule of $\mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev}$ with basis $\mathcal{L}^\#$ for various $(\mathcal{O}', \mathcal{L}')$ in $\mathcal{I}(\tilde{\mathfrak{l}}_\eta^\phi)$. By [L4, 21.1(c)], for any $j \in J$, we have

$$\text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta]) \in \mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev}.$$

Hence $F_0 \in \mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev}$, $F_1 \in v\mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev}$. We have clearly $h\mathcal{L}_1^\# \in \mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev}$. Since $v\mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev} \cap \mathcal{K}(\tilde{\mathfrak{l}}_\eta^\phi)^{ev} = 0$, we deduce from (c) that $h\mathcal{L}_1^\# = F_0$; that is,

$$h\mathcal{L}_1^\# = \sum_{j \in J} h'_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta])),$$

where $h'_j \in \mathbf{Z}[v^2, v^{-2}]$. Applying ${}^e\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}$ and using the transitivity property 4.2 from [LY] (and also 10.4(a)) we obtain

$$hI = \sum_{j \in J} h'_j I_{\varpi_j},$$

where $\varpi_j \in \mathbf{E}'$ for $j \in J$. Since $\gamma^{-1}(\sum_{j \in J} h'_j I_{\varpi_j})$ is fixed by ${}^\heartsuit$, it follows that hI' is fixed by ${}^\heartsuit$. Since $h \neq 0$ and $h^\heartsuit = h$, we deduce that $I'^\heartsuit = I'$. Using (a) and the induction hypothesis, we see that

$$\begin{aligned} (v^{-\dim \mathcal{O} b})^\heartsuit + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'}^\heartsuit (-1)^{\dim \mathcal{O}' b'} \\ = v^{-\dim \mathcal{O} b} + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'} b'. \end{aligned}$$

Using this and 14.6(b), we deduce that $(-v)^{-\dim \mathcal{O}} s_b \tilde{b} - v^{-\dim \mathcal{O} b}$ is a \mathcal{A} -linear combination of elements in $\cup_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \mathbf{B}_{\mathcal{O}'}$.

Since $b \in \mathbf{B}_{\mathcal{O}}$ and $\tilde{b} \in \mathbf{B}$ this forces $\tilde{b} = b$ and $s_b = (-1)^{\dim \mathcal{O}}$. This completes the proof of the lemma.

14.8. We show:

(a) *Let $\mathcal{O}, \mathcal{O}'$ be G_0 -orbits in $\mathfrak{g}_\delta^{nil}$ and let B, B' in \mathfrak{B} be such that the support of B (resp. B') is the closure of \mathcal{O} (resp. \mathcal{O}'). We have $(B : B')^\heartsuit = (-1)^{\dim \mathcal{O} + \dim \mathcal{O}'} (B : B')$.*

If $\psi(B) \neq \psi(B')$, then $(B : B') = 0$ by [LY, 7.9(a)] and (a) holds. Assume now that $\psi(B) = \psi(B')$. We can assume that $\psi(B) = \psi(B') = \xi$. It is enough to prove that, if $b \in \mathbf{B}_{\mathcal{O}}$, $b' \in \mathbf{B}_{\mathcal{O}'}$, then $(b : b')^\heartsuit = (-1)^{\dim \mathcal{O} + \dim \mathcal{O}'} (b : b')$. Using 14.2 and 14.7, we have

$$(b : b')^\heartsuit = (b^\heartsuit : b'^\heartsuit) = ((-1)^{\dim \mathcal{O} b}, (-1)^{\dim \mathcal{O}' b'})$$

and (a) is proved.

14.9. We show:

(a) *For any B, B' in \mathfrak{B} we have $P_{B, B'} \in \mathbf{N}[v^{-2}]$, $\tilde{P}_{B, B'} \in \mathbf{N}[v^{-2}]$, $\tilde{\Lambda}_{B, B'} \in \mathbf{Q}((v^2))$ (notation of 13.6).*

The proof is similar to that of 13.7(a). With the notation in 14.8(a) we apply ${}^\heartsuit$ to

$$v^{-\dim \mathcal{O} - \dim \mathcal{O}'} (B : B') = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \tilde{\Lambda}_{B_1, B_2} \tilde{P}_{B_2, B'}$$

(see 13.6(d)); using 14.8 we obtain

$$v^{-\dim \mathcal{O} - \dim \mathcal{O}'} (B : B') = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B}^\heartsuit \tilde{\Lambda}_{B_1, B_2}^\heartsuit \tilde{P}_{B_2, B'}^\heartsuit.$$

When B, B' vary, this can be expressed as the decomposition of the matrix $\mathcal{M} = (v^{c_B - c_{B'}} v^{-\kappa(B) - \kappa(B')}) (B : B')$ (indexed by $\mathfrak{B} \times \mathfrak{B}$) as a product of three matrices $\mathcal{S}^\heartsuit \mathcal{T}^\heartsuit \mathcal{S}'^\heartsuit$ where \mathcal{S}^\heartsuit (resp. \mathcal{S}'^\heartsuit) is the matrix indexed by $\mathfrak{B} \times \mathfrak{B}$ whose (B, B') -entry is $P_{B', B}^\heartsuit$ (resp. $\tilde{P}_{B, B'}^\heartsuit$) and \mathcal{T}^\heartsuit is the matrix indexed by $\mathfrak{B} \times \mathfrak{B}$ whose (B, B') -entry is $\tilde{\Lambda}_{B, B'}^\heartsuit$. Recall from 13.6 that we have also $\mathcal{M} = \mathcal{S} \mathcal{T} \mathcal{S}'$ (notation of 13.6). Thus we have

$$\mathcal{S}^\heartsuit \mathcal{T}^\heartsuit \mathcal{S}'^\heartsuit = \mathcal{S} \mathcal{T} \mathcal{S}'.$$

Now by 13.6(c),(e), the matrix \mathcal{T} (and hence the matrix \mathcal{T}^\heartsuit) belongs to a subgroup of GL_N ($N = \sharp(\mathfrak{B})$) of the form $GL_{N_1} \times \dots \times GL_{N_k}$ where N_1, \dots, N_k are the sizes of the various subsets \mathfrak{B}_O ; moreover, by 13.6(a),(b), the matrix \mathcal{S} (hence the matrix \mathcal{S}^\heartsuit) belongs to the unipotent radical of a parabolic subgroup of GL_N with Levi subgroup equal to the subgroup of GL_N considered above, while the matrix \mathcal{S}' (hence the matrix \mathcal{S}'^\heartsuit) belongs to the unipotent radical of the opposite parabolic subgroup with the same Levi subgroup. This forces the equalities $\mathcal{S}^\heartsuit = \mathcal{S}$, $\mathcal{T}^\heartsuit = \mathcal{T}$, $\mathcal{S}'^\heartsuit = \mathcal{S}'$. Now the equality $\mathcal{S}^\heartsuit = \mathcal{S}$ implies $P_{B',B}^\heartsuit = P_{B',B}$ for all B', B in \mathfrak{B} . Similarly, from $\mathcal{T}^\heartsuit = \mathcal{T}$ we see that $\tilde{\Lambda}_{B,B'}^\heartsuit = \tilde{\Lambda}_{B,B'}$ for all B, B' in \mathfrak{B} and from $\mathcal{S}'^\heartsuit = \mathcal{S}'$ we see that $\tilde{P}_{B,B'}^\heartsuit = \tilde{P}_{B,B'}$. This proves (a).

Theorem 14.10.

(a) Let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ and let $A = \mathcal{L}^\sharp \in \mathcal{Q}(\mathfrak{g}_\delta)$. We have $\mathcal{H}^a A = 0$ for any odd integer a .

(b) Let $\varpi \in \mathbf{E}'$. We have $\mathcal{H}^a(I_\varpi) = 0$ for any odd integer a .

(c) Let $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$ be as in 4.1(a) of [LY] with $\epsilon = \dot{\eta}$, and let $(\mathcal{O}', \mathcal{L}') \in \mathcal{I}(\mathfrak{l}_\eta)$ (see 1.2). We form $A' = \mathcal{L}'^\sharp \in \mathcal{Q}(\mathfrak{l}_\eta)$. We have $\mathcal{H}^a({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')) = 0$ for any odd integer a .

(d) In the setup of (c), ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')$ is a direct sum of complexes of the form $\mathcal{L}^\sharp[s]$ for various $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ and various even integers s .

(a) follows from 14.9(a).

We prove (b). Let $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$ be such that $\varpi \in \mathbf{c}$. In \mathbf{V} we have $T_{\mathbf{c}} = \oplus_{b \in \mathbf{B}} f_b v^{-\kappa(b)} b$, where $f_b \in \mathcal{A}$. Applying ${}^\heartsuit$ and using $T_{\mathbf{c}}^\heartsuit = T_{\mathbf{c}}$ and $(v^{-\kappa(b)} b)^\heartsuit = v^{-\kappa(b)} b$ for $b \in \mathbf{B}$ (see 14.7) we see that

$$\oplus_{b \in \mathbf{B}} f_b^\heartsuit v^{-\kappa(b)} b = \oplus_{b \in \mathbf{B}} f_b v^{-\kappa(b)} b.$$

Hence $f_b^\heartsuit = f_b$, that is, $f_b \in \mathbf{Z}[v^2, v^{-2}]$. Thus,

(e) I_ϖ is isomorphic to a direct sum of complexes of the form $B[-\kappa(B)][2s]$ with $B \in \mathfrak{B}$ and $s \in \mathbf{Z}$.

Hence $\mathcal{H}^a I_\varpi$ is isomorphic to a direct sum of sheaves of the form $\mathcal{H}^{a+2s}(B[-\kappa(B)])$ with $B \in \mathfrak{B}$. Hence the desired result follows from (a).

We prove (c). By 1.5(a) of [LY] we can find $\mathfrak{q}_*, (\hat{\mathfrak{p}}_*, \hat{L}, \hat{P}_0, \hat{\mathfrak{l}}, \hat{\mathfrak{l}}_*, \hat{\mathfrak{u}}_*)$ as in 4.2 of [LY] with $\epsilon = \dot{\eta}$ and a cuspidal perverse sheaf C in $\mathcal{Q}(\hat{\mathfrak{l}}_\eta)$ such that $A'[s']$ is a direct summand of $\text{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(C[-\dim \mathfrak{l}_\eta])$ for some $s' \in \mathbf{Z}$ hence, by 4.2(a) of [LY], ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')[s']$ is a direct summand of ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C[-\dim \mathfrak{l}_\eta])$; moreover, by [L4, 21.1(c)], we have $s' = 2s''$ for some $s'' \in \mathbf{Z}$. Hence $\mathcal{H}^a({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A'))$ is a direct summand of

$$\mathcal{H}^{a-2s''}({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C[-\dim \mathfrak{l}_\eta])).$$

We can assume that $\hat{\mathfrak{p}}_\eta$ is an ϵ -spiral with splitting \mathfrak{m}_* (in 10.1) and that $C = \tilde{C}$ (in 10.1). Then ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C[-\dim \mathfrak{l}_\eta]) = I_\varpi$ for some $\varpi \in \mathbf{E}'$ and $\mathcal{H}^a({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A'))$ is a direct summand of $\mathcal{H}^{a-2s''} I_\varpi$. The desired result follows from (b).

We prove (d). As in the proof of (c), ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')$ is a direct summand of $I_\varpi[-2s'']$ for some $\varpi \in \mathbf{E}'$ and $s'' \in \mathbf{Z}$. This, together with (e) gives the desired result. The theorem is proved.

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